**G12 Mathematics Extended Part**

**Module 1 – Calculus and Statistics**

**Mock Examination** **Date: January 2023**

**Marking Scheme**

**Section A (50 marks)**

1. (a) Since $E(X^2) = 4.3$,

$$(-2)^2(0.1) + (0)^2(0.2) + (1)^2(0.3) + a^2(0.4) = 4.3 \quad \text{1M}$$
$$0.4a^2 + 0.7 = 4.3$$
$$a^2 = 9$$
$$a = \pm 3$$

Since $a > 0$, $a = 3$. 1A

1. (b) Method 1

$E(X)$
$= (-2)(0.1) + (0)(0.2) + (1)(0.3) + (3)(0.4)$
$= 1.3$

$E(3-2X)$
$= 3 - 2E(X)$ *
$= 3 - 2(1.3)$
$= 0.4$ 1A

$Var(X)$
$= E(X^2) - (E(X))^2$
$= 4.3 - 1.3^2$ 1M
$= 2.61$

$Var(3-2X)$
$= (-2)^2 Var(X)$ *
$= 4(2.61)$
$= 10.44$ 1A

*1M for either

Method 2

$E(3-2X)$

$=[3-2(-2)](0.1)+[3-2(0)](0.2)+[3-2(1)](0.3)+[3-2(3)](0.4)$ **

1A

$=0.4$

Method 2.1

$E((3-2X)^2)$

$=[3-2(-2)]^2(0.1)+[3-2(0)]^2(0.2)+[3-2(1)]^2(0.3)+[3-2(3)]^2(0.4)$

**

$=10.6$

$Var(3-2X)$

$=E((3-2X)^2)-(E(3-2X))^2$

$=10.6-0.4^2$ 1M

$=10.44$ 1A

**1M for either

Method 2.2

$Var(3-2X)$

$=E([(3-2X)-E(3-2X)]^2)$

$=E([(3-2X)-0.4]^2)$ 1M

$=E((2.6-2X)^2)$

$=[2.6-2(-2)]^2(0.1)+[2.6-2(0)]^2(0.2)+[2.6-2(1)]^2(0.3)+[2.6-2(3)]^2(0.4)$

**

$=10.44$ 1A

**1M for either

2. (a) $\mathrm{P}(A \cap B)$

$= \mathrm{P}(A \mid B) \cdot \mathrm{P}(B)$

$= \frac{1}{6}\mathrm{P}(B)$ 1A

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\frac{9}{20} = \frac{1}{5} + \mathrm{P}(B) - \frac{1}{6}\mathrm{P}(B)$ 1M

$\frac{1}{4} = \frac{5}{6}\mathrm{P}(B)$

$\mathrm{P}(B) = \frac{3}{10}$ 1A

2. (b) Method 1

$\mathrm{P}(A' \cap B)$

$= \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= \frac{3}{10} - \frac{1}{6}\left(\frac{3}{10}\right)$ 1M

$= \frac{1}{4}$ 1A

Method 2

$\mathrm{P}(A' \cap B)$

$= \mathrm{P}(A \cup B) - \mathrm{P}(A)$

$= \frac{9}{20} - \frac{1}{5}$ 1M

$= \frac{1}{4}$ 1A

2. (c) Method 1

Note that $A' \cap B' = (A \cup B)'$.

$$\begin{aligned} & \mathrm{P}(A' \cap B') \\ &= 1 - \mathrm{P}(A \cup B) \\ &= 1 - \frac{9}{20} \\ &= \frac{11}{20} \end{aligned}$$

Method 2

$$\begin{aligned} & \mathrm{P}(A' \cap B') \\ &= \mathrm{P}(A') - \mathrm{P}(A' \cap B) \\ &= \left(1 - \frac{1}{5}\right) - \frac{1}{4} \\ &= \frac{11}{20} \end{aligned}$$

Since $\mathrm{P}(A' \cap B') \neq 0$, $A'$ and $B'$ are not mutually exclusive.

1A f.t.

3. (a) Since $P(X < 10) = 0.8413$,

$$P\left(Z < \frac{10-\mu}{2}\right) = 0.8413$$ 1M

$$P\left(0 < Z < \frac{10-\mu}{2}\right) = 0.8413 - 0.5$$

$$P\left(0 < Z < \frac{10-\mu}{2}\right) = 0.3413$$

$$\frac{10-\mu}{2} = 1$$

$$\mu = 8$$ 1A

3. (b) (i) $P(A)$

$= P(5 \le X \le 6)$

$= P\left(\frac{5-8}{2} \le Z \le \frac{6-8}{2}\right)$

$= P(-1.5 \le Z \le -1)$

$= 0.4332 - 0.3413$

$= 0.0919$ 1A

3. (b) (ii) $P(B_t)$

$= P(-1 < X - t < 1)$

$= P(t - 1 < X < t + 1)$

Since $X$ is a normal random variable, $P(B_t)$ attains its greatest value at $t = \mu = 8$ and $T = 8$. 1M

Then $B_T$ is the event that

$$T - 1 < X < T + 1$$

$$8 - 1 < X < 8 + 1$$

$$7 < X < 9$$

It follows that $A \cap B_T = \varnothing$ and $P(A \cap B_T) = 0$. 1M

Note that $P(A) > 0$ and $P(B_T) > 0$.

Since $P(A \cap B_T) \ne P(A)P(B_T)$, $A$ and $B_T$ are not independent.

1A f.t.

4. (a) Method 1

The required probability

$= (0.6)(1-0.7) + (0.3)(1-0.8) + (0.1)(1-0.9)$ 1M

$= 0.25$ 1A

Method 2

The required probability

$= 1 - [(0.6)(0.7) + (0.3)(0.8) + (0.1)(0.9)]$ 1M

$= 0.25$ 1A

4. (b) (i) The required probability

$= C_3^5(0.25)^3(1-0.25)^2 + C_4^5(0.25)^4(1-0.25) + (0.25)^5$ 1M

$= \frac{53}{512}$ 1A

$= 0.103515625$

$= 0.1035$ (cor. to 4 d. p.)

4. (b) (ii) The probability that Harrods is late for school on at least 3 consecutive days in the week given that he is late for school on at least 3 days in the week

$= \frac{(3)(0.25)^3(1-0.25)^2 + (4)(0.25)^4(1-0.25) + (0.25)^5}{\frac{53}{512}}$ 1M for numerator + 1M for denominator

$= \frac{20}{53}$

$\approx 0.37735849$

$< 0.4$

The claim is disagreed. 1A f.t.

5. (a) $(4-e^{-px})^3$

$= 4^3 - C_1^3(4)^2(e^{-px}) + C_2^3(4)(e^{-px})^2 - (e^{-px})^3$ 1M

$= 64 - 48e^{-px} + 12e^{-2px} - e^{-3px}$

$= 64 - 48\left[1 + (-px) + \frac{(-px)^2}{2!} + \frac{(-px)^3}{3!} + \cdots\right]$

$+ 12\left[1 + (-2px) + \frac{(-2px)^2}{2!} + \frac{(-2px)^3}{3!} + \cdots\right]$ 1M

$- \left[1 + (-3px) + \frac{(-3px)^2}{2!} + \frac{(-3px)^3}{3!} + \cdots\right]$

$= 27 + 27px - \frac{9p^2}{2}x^2 - \frac{7p^3}{2}x^3 + \cdots$ 1A

5. (b)

$$\left(-\frac{9p^2}{2}\right) + \left(-\frac{7p^3}{2}\right) = -8$$

$$7p^3 + 9p^2 - 16 = 0$$

$$(p-1)(7p^2 + 16p + 16) = 0$$

$$p = 1 \text{ or } 7p^2 + 16p + 16 = 0$$

Consider $7p^2 + 16p + 16 = 0$.

$\Delta$

$= 16^2 - 4(7)(16)$

$= -192$ 1M

$< 0$

It follows that $7p^2 + 16p + 16 = 0$ has no real roots.

Thus $p = 1$. 1A

6. (a) Since $r$ is an integer, $x^{2r}=(x^2)^r$ for all non-zero real values of $x$.

Then $g(x)=\dfrac{\ln x^{2r}}{\ln x^2+1}=\dfrac{\ln(x^2)^r}{\ln x^2+1}=\dfrac{r\ln x^2}{\ln x^2+1}$ for all non-zero real values of $x$.

$$\lim_{x\to\infty} g(x)=3$$

$$\lim_{x\to\infty}\frac{r\ln x^2}{\ln x^2+1}=3$$

$$\lim_{x\to\infty}\frac{\frac{r\ln x^2}{\ln x^2}}{\frac{\ln x^2+1}{\ln x^2}}=3$$

$$\lim_{x\to\infty}\frac{r}{1+\frac{1}{\ln x^2}}=3 \qquad \text{1M}$$

$$\frac{r}{1+0}=3$$

$$r=3 \qquad 1$$

6. (b) Note that $g(x)=\dfrac{3\ln x^2}{\ln x^2+1}$ for all non-zero real values of $x$.

$g'(x)$

$$=\frac{3\left[\left(\frac{2x}{x^2}\right)(\ln x^2+1)-(\ln x^2)\left(\frac{2x}{x^2}\right)\right]}{(\ln x^2+1)^2} \qquad \text{1M}$$

$$=\frac{6}{x(\ln x^2+1)^2}$$

The equation of the tangent to the curve $y=g(x)$ at $x=e$ is

$$\frac{y-g(e)}{x-e}=g'(e)$$

$$\frac{y-\frac{3\ln e^2}{\ln e^2+1}}{x-e}=\frac{6}{e(\ln e^2+1)^2} \qquad \text{1M}$$

$$\frac{y-2}{x-e}=\frac{2}{3e}$$

$$3ey-6e=2x-2e$$

$$2x-3ey+4e=0 \qquad \text{1A}$$

7. (a) $h'(x)$

$= 2x + 14 - \frac{72}{x^2}$ 1M

Put $h'(x) = 0$.

$$2x + 14 - \frac{72}{x^2} = 0$$

$$x^3 + 7x^2 - 36 = 0$$

$$(x-2)(x^2 + 9x + 18) = 0$$ 1M

$$(x-2)(x+3)(x+6) = 0$$

$$x = 2 \text{ (rej.) or } -3 \text{ or } -6$$

Method 1 [First Derivative Test]

| $x$ | $x=-7$ | $-7<x<-6$ | $x=-6$ | $-6<x<-3$ | $x=-3$ |
|---|---|---|---|---|---|
| $h(x)$ | $-\frac{415}{7}$ | | $-60$ | | $-57$ |
| $h'(x)$ | $-$ | $-$ | 0 | $+$ | 0 |

| $x$ | $-3<x<-1$ | $x=-1$ |
|---|---|---|
| $h(x)$ | | $-85$ |
| $h'(x)$ | $-$ | $-$ |

Method 2 [Second Derivative Test]

$h''(x)$

$= 2 - 72\left(-\frac{2}{x^3}\right)$

$= 2 + \frac{144}{x^3}$

$h''(-6)$

$= 2 + \frac{144}{(-6)^3}$

$= \frac{4}{3}$

$> 0$

$h''(-3)$

$= 2 + \frac{144}{(-3)^3}$

$= -\frac{10}{3}$

$< 0$

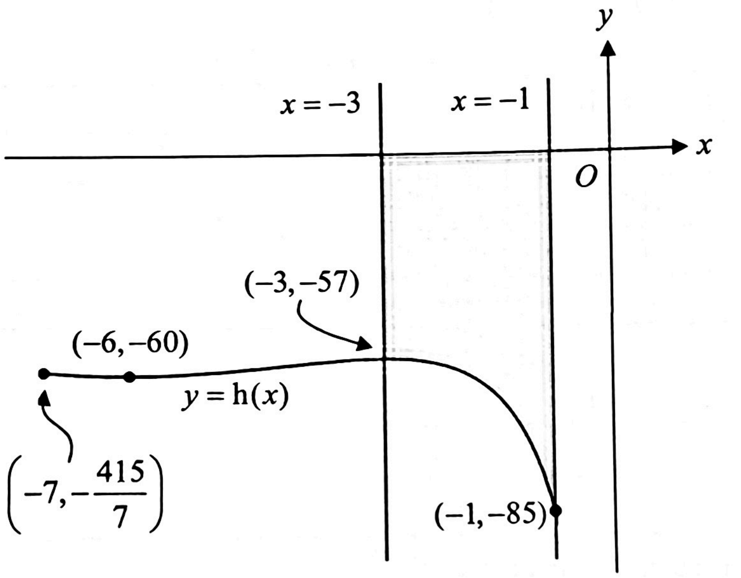
Note that $h(-7)=-\frac{415}{7}$, $h(-6)=-60$, $h(-3)=-57$ and $h(-1)=-85$.

1M for first/second derivative test

The greatest value of $h(x)$ is attained at $x=-3$ and the least value of $h(x)$ is attained at $x=-1$.

Thus $\alpha=-3$ and $\beta=-1$. 1A for both

7. (b)




The required area

$$=-\int_{\alpha}^{\beta} h(x)\,dx$$

$$=-\int_{-3}^{-1}\left(x^2+14x+\frac{72}{x}\right)dx$$ 1M for sign and limits

$$=-\left[\frac{1}{3}x^3+14\left(\frac{1}{2}x^2\right)+72\ln|x|\right]_{-3}^{-1}$$ 1M

$$=-\left[\frac{1}{3}x^3+7x^2+72\ln|x|\right]_{-3}^{-1}$$

$$=\frac{142}{3}+72\ln 3$$ 1A

8. (a) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2e^{kx^2})$

$=(2x)(e^{kx^2})+(x^2)(2kxe^{kx^2})$ 1M

$=2xe^{kx^2}+2kx^3e^{kx^2}$

Let $u=kx^2$. Then $\mathrm{d}u=2kx\,\mathrm{d}x$.

It follows that

$$2kx^3e^{kx^2}=\frac{\mathrm{d}}{\mathrm{d}x}(x^2e^{kx^2})-2xe^{kx^2}$$

$$\int 2kx^3e^{kx^2}\,\mathrm{d}x=x^2e^{kx^2}-\int 2xe^{kx^2}\,\mathrm{d}x$$ 1M

$$\int 2kx^3e^{kx^2}\,\mathrm{d}x=x^2e^{kx^2}-\frac{1}{k}\int e^u\,\mathrm{d}u$$ 1M

$$2k\int x^3e^{kx^2}\,\mathrm{d}x=x^2e^{kx^2}-\frac{1}{k}e^u+C'$$

$$\int x^3e^{kx^2}\,\mathrm{d}x=\frac{1}{2k}x^2e^{kx^2}-\frac{1}{2k^2}e^{kx^2}+C$$ 1A

8. (b) (i) Since $\frac{\mathrm{d}y}{\mathrm{d}x}=x^3e^{kx^2}$,

$$y=\int x^3e^{kx^2}\,\mathrm{d}x$$

$$y=\frac{1}{2k}x^2e^{kx^2}-\frac{1}{2k^2}e^{kx^2}+C$$

Put $\left(0,1-\frac{1}{2k^2}\right)$ into the equation.

$$1-\frac{1}{2k^2}=\frac{1}{2k}(0)^2e^{k(0)^2}-\frac{1}{2k^2}e^{k(0)^2}+C$$ 1M

$$C=1$$

It follows that

$$y=\frac{1}{2k}x^2e^{kx^2}-\frac{1}{2k^2}e^{kx^2}+1$$

$$y-1=\frac{e^{kx^2}}{2k^2}(kx^2-1)$$

$$\frac{y-1}{kx^2-1}=\frac{e^{kx^2}}{2k^2}$$

$$\ln\left(\frac{y-1}{kx^2-1}\right)=\ln\left(\frac{e^{kx^2}}{2k^2}\right)$$

$$\ln\left(\frac{y-1}{kx^2-1}\right)=kx^2-\ln(2k^2)$$ 1A

8. (b) (ii) Note that the slope is $k = \frac{1}{e^2}$.

Method 1

When $\ln\left(\frac{y-1}{kx^2-1}\right) = 0$,

1M

$$kx^2 - \ln(2k^2) = 0$$

$$\frac{x^2}{e^2} - \ln\left(\frac{2}{(e^2)^2}\right) = 0$$

$$\frac{x^2}{e^2} - \ln\left(\frac{2}{e^4}\right) = 0$$

$$x^2 = -(4 - \ln 2)e^2$$

This contradicts the fact that $x^2 \geq 0$ for all real values of $x$.

Method 2

$\ln\left(\frac{y-1}{kx^2-1}\right)$

$= kx^2 - \ln(2k^2)$

$= \frac{x^2}{e^2} - \ln\left(\frac{2}{(e^2)^2}\right)$

$= \frac{x^2}{e^2} - \ln\left(\frac{2}{e^4}\right)$

$\geq -\ln\left(\frac{2}{e^4}\right)$ $\left[\because \frac{x^2}{e^2} \geq 0\right]$ 1M

$= 4 - \ln 2$

$> 0$

It follows that the graph does not have any intercept on the horizontal axis.

The claim is incorrect. 1A f.t.

**Section B (50 marks)**

9. (a) The required probability

$=\frac{e^{-1}1^0}{0!}+\frac{e^{-1}1^1}{1!}+\frac{e^{-1}1^2}{2!}$ 1M for Poisson probability

$=2.5e^{-1}$ 1A

$\approx 0.919698602$

$=0.9197$ (cor. to 4 d. p.)

9. (b) (i) Note that the mean number of mistakes introduced by the secretary to each page of any document is $\lambda$ and the mean number of mistakes introduced by the secretary to any document is $5\lambda$.

$$\frac{\left(\frac{e^{-1}1^2}{2!}\right)\left(\frac{e^{-5\lambda}(5\lambda)^0}{0!}\right)}{\left(\frac{e^{-1}1^0}{0!}\right)\left(\frac{e^{-5\lambda}(5\lambda)^2}{2!}\right)+\left(\frac{e^{-1}1^1}{1!}\right)\left(\frac{e^{-5\lambda}(5\lambda)^1}{1!}\right)+\left(\frac{e^{-1}1^2}{2!}\right)\left(\frac{e^{-5\lambda}(5\lambda)^0}{0!}\right)}=0.16$$

1M for numerator + 1M for denominator

Award even if $\lambda$ is used in place of $5\lambda$

$$\frac{\frac{1}{2}}{\frac{25}{2}\lambda^2+5\lambda+\frac{1}{2}}=0.16$$

$$\frac{1}{25\lambda^2+10\lambda+1}=0.16$$

$$25\lambda^2+10\lambda-5.25=0$$

$$\lambda^2+0.4\lambda-0.21=0$$

$$(\lambda-0.3)(\lambda+0.7)=0$$

$$\lambda=0.3 \text{ or } -0.7$$

Since $\lambda\geq 0$, $\lambda=0.3$. 1

9. (b) (ii) (1) The probability that a document prepared by Noah and processed by his secretary contains no mistakes

$$= \left(\frac{e^{-1}1^0}{0!}\right)\left(\frac{e^{-1.5}1.5^0}{0!}\right)$$

$$= e^{-2.5}$$

The probability that a document prepared by Noah and processed by his secretary contains exactly 1 mistake

$$= \left(\frac{e^{-1}1^1}{1!}\right)\left(\frac{e^{-1.5}1.5^0}{0!}\right) + \left(\frac{e^{-1}1^0}{0!}\right)\left(\frac{e^{-1.5}1.5^1}{1!}\right)$$

$$= 2.5e^{-2.5}$$

The probability that a document prepared by Noah and processed by his secretary contains exactly 2 mistakes

$$= \left(\frac{e^{-1}1^2}{2!}\right)\left(\frac{e^{-1.5}1.5^0}{0!}\right) + \left(\frac{e^{-1}1^1}{1!}\right)\left(\frac{e^{-1.5}1.5^1}{1!}\right) + \left(\frac{e^{-1}1^0}{0!}\right)\left(\frac{e^{-1.5}1.5^2}{2!}\right)$$

$$= 3.125e^{-2.5}$$

The probability that a document prepared by Noah and processed by his secretary is *accurate*

$$= e^{-2.5} + 2.5e^{-2.5} + 3.125e^{-2.5}$$

$$= 6.625e^{-2.5}$$

The required expected number of mistakes

$$= (0)\left(\frac{e^{-2.5}}{6.625e^{-2.5}}\right) + (1)\left(\frac{2.5e^{-2.5}}{6.625e^{-2.5}}\right) + (2)\left(\frac{3.125e^{-2.5}}{6.625e^{-2.5}}\right)$$

1M for conditional probability + 1M for expected value

$= \frac{70}{53}$ or 1.3208 (cor. to 4 d. p.) 1A

9. (b) (ii) (2) The required probability

$$= [C_2^5(6.625e^{-2.5})^2(1-6.625e^{-2.5})^3(6.625e^{-2.5})][C_2^4(6.625e^{-2.5})^2(1-6.625e^{-2.5})^2]$$

1M for first factor + 1M for second factor

$\approx 0.056378512$

$= 0.0564$ (cor. to 4 d. p.) 1A

10. (a) (i) A 95% confidence interval for $\mu$

$$=\left(35.2-1.96\left(\frac{2.5}{\sqrt{64}}\right),35.2+1.96\left(\frac{2.5}{\sqrt{64}}\right)\right)$$ 1M + 1A

$$=(34.5875,35.8125)$$ 1A

10. (a) (ii) Let $z$ be the critical value of the new confidence interval.

$$2z\left(\frac{2.5}{\sqrt{64}}\right)\leq 2(1.96)\left(\frac{2.5}{\sqrt{64}}\right)(1-20\%)$$ 1M

$$z\leq 1.568$$

The confidence level of the new confidence interval

$\leq 2(0.4406)(100\%)$ 1M

$=88.12\%$

Thus $0<k\leq 88.12$. 1A

10. (b) (i) Let $b$ cm$^2$ be the sum of squares of the deviations of the lengths of the rods in sample $B$ from the mean of the lengths of the rods in sample $B$.

$$\frac{b}{n}=\left(\frac{b}{n-1}\right)(1-2\%)$$ 1M

$$\frac{n-1}{n}=0.98$$

$$n=50$$ 1

10. (b) (ii) (1) $\bar{x}$ 1M

$= \frac{(35.2)(64)+1840}{64+50}$

$\approx 35.90175439$ *

$= 35.9018$ (cor. to 4 d. p.)

Let $a$ cm$^2$ be the sum of squares of the lengths of the rods in sample $A$.

$$2.1^2 = \frac{a-(64)(35.2)^2}{64-1} \quad **$$

$$a = 79576.39$$

The sum of squares of the lengths of the rods in sample $A$ is 79576.39 cm$^2$.

$s$

$\approx \sqrt{\frac{(79576.39+68018.25)-(64+50)(35.90175439)^2}{(64+50)-1}}$ **

$\approx 2.409310203$

$= 2.4093$ (cor. to 4 d. p.) *

**1M for either + *1A for both

10. (b) (ii) (2) Let $\overline{X}$ cm be the mean of the lengths of the rods in sample $D$.

By the central limit theorem, $\overline{X} \sim \mathrm{N}\left(\bar{x}, \frac{s^2}{70}\right)$ approximately.

1M

The required probability

$= \mathrm{P}(70\overline{X} > 2500)$ 1A

$\approx \mathrm{P}\left(Z > \frac{\frac{2500}{70}-\bar{x}}{\frac{s}{\sqrt{70}}}\right)$

$\approx \mathrm{P}\left(Z > \frac{\frac{2500}{70}-35.90175439}{\frac{2.409310203}{\sqrt{70}}}\right)$

$\approx \mathrm{P}(Z > -0.651006016)$

$\approx 0.5+0.2422$

$= 0.7422$ 1A

11. (a) $f'(x)$

$= \frac{(6x^5)(x^3-1)-(x^6)(3x^2)}{(x^3-1)^2}$ 1M

$= \frac{3x^5(x^3-2)}{(x^3-1)^2}$ 1A

$= \frac{3x^8-6x^5}{(x^3-1)^2}$

$f''(x)$

$= \frac{[3(8x^7)-6(5x^4)](x^3-1)^2-(3x^8-6x^5)[2(x^3-1)(3x^2)]}{(x^3-1)^4}$ 1M

$= \frac{6x^4(x^3-1)[(4x^3-5)(x^3-1)-(3x^6-6x^3)]}{(x^3-1)^4}$

$= \frac{6x^4(x^6-3x^3+5)}{(x^3-1)^3}$ 1A

$= \frac{6x^{10}-18x^7+30x^4}{(x^3-1)^3}$

11. (b) (i) $I$

$= \int_{-2}^{0} f(x)\,dx$

$\approx \frac{1}{2}\left(\frac{0-(-2)}{5}\right)\{f(-2)+2[f(-1.6)+f(-1.2)+f(-0.8)+f(-0.4)]+f(0)\}$

1M

$\approx -3.247832838$

$= -3.2478$ (cor. to 4 d. p.) 1A

11. (b) (ii)

<u>Method 1</u>

$J$

$= \int_{-2}^{0} \frac{4x^3-2}{x^3-1}\,dx$

$= -2\int_{-2}^{0} \frac{-2x^3+1}{x^3-1}\,dx$

$= -2\int_{-2}^{0} \frac{(x^6-2x^3+1)-x^6}{x^3-1}\,dx$ 1M

$= -2\int_{-2}^{0} \frac{(x^3-1)^2-x^6}{x^3-1}\,dx$

Method 2

Note that $\frac{x^6}{x^3-1}=(x^3+1)+\frac{1}{x^3-1}$ and $\frac{1}{x^3-1}=\frac{x^6}{x^3-1}-(x^3+1)$ .

$\frac{4x^3-2}{x^3-1}$

$=4+\frac{2}{x^3-1}$

$=4+2\left[\frac{x^6}{x^3-1}-(x^3+1)\right]$ 1M

$=2-2x^3+\frac{2x^6}{x^3-1}$

$J$

$=\int_{-2}^{0}\frac{4x^3-2}{x^3-1}\,dx$

$=\int_{-2}^{0}\left(2-2x^3+\frac{2x^6}{x^3-1}\right)dx$

$=-2\int_{-2}^{0}(x^3-1)\,dx+2\int_{-2}^{0}\frac{x^6}{x^3-1}\,dx$ 1M

$=-2\left[\frac{1}{4}x^4-x\right]_{-2}^{0}+2I$ 1M

$\approx -2(-6)+2(-3.247832838)$

$\approx 5.504334324$

$=5.5043$ (cor. to 4 d. p.) 1A

11. (b) (iii) Note that $f''(x)<0$ for all $-2\le x<0$ .

The estimate in (b)(i) is an under-estimate. 1

From (b)(ii),

$J=12+2I$

$J>12+2(-3.247832838)$

$J>5.504334324$

$J>5.5$

The claim is correct. 1A f.t.

12. (a) Since $T \geq 0$ and $k > 1$, $T+k > 1$ and $\ln(T+k) > 0$.

$$\frac{d^2v}{dt^2}$$

$$= \frac{300\left\{\left[[2\ln(t+k)]\left(\frac{1}{t+k}\right)\right](t+k) - [\ln(t+k)]^2(1)\right\}}{(t+k)^2}$$ 1M + 1A

$$= \frac{300\ln(t+k)[2-\ln(t+k)]}{(t+k)^2}$$

Since the rate of change of the speed of the particle is maximum $T$ s after the start of the experiment,

$$\left.\frac{d^2v}{dt^2}\right|_{t=T} = 0$$

$$\frac{300\ln(T+k)[2-\ln(T+k)]}{(T+k)^2} = 0$$ 1M

$$2 - \ln(T+k) = 0$$

$$\ln(T+k) = 2$$

$$T+k = e^2$$

$$T = e^2 - k$$ 1A

Remark: As the existence of a maximum value of $\frac{dv}{dt}$ is stated in the question and $\left.\frac{d^2v}{dt^2}\right|_{t=T} = 0$ gives rise to only one possible value of $T$, testing for maximum is not necessary.

12. (b) (i) Method 1

Let $u = \ln(t+k)$. Then $du = \frac{1}{t+k}dt$. 1M

When $t = 0$, $u = \ln k$ and $v = 100(\ln 2)^3$.
When $t = T$, $u = \ln(T+k)$ and $v = 800$.

$$800 - 100(\ln 2)^3 = \int_0^T \frac{300[\ln(t+k)]^2}{t+k}dt$$ 1M

$$800 - 100(\ln 2)^3 = 300\int_{\ln k}^{\ln(T+k)} u^2\,du$$

$$8 - (\ln 2)^3 = 3\left[\frac{1}{3}u^3\right]_{\ln k}^{2}$$ 1M

$$8 - (\ln 2)^3 = 8 - (\ln k)^3$$

$$k = 2$$ 1

Method 2 1M

Let $u=\ln(t+k)$. Then $du=\dfrac{1}{t+k}dt$.

$v$

$$=\int\frac{300[\ln(t+k)]^2}{t+k}dt$$

$$=300\int u^2\,du$$

$$=300\left(\frac{1}{3}u^3\right)+C \qquad \text{1M}$$

$$=100[\ln(t+k)]^3+C$$

When $t=T$, $v=800$.

$$100[\ln(T+k)]^3+C=800 \qquad *$$

$$100(2)^3+C=800$$

$$C=0$$

When $t=0$, $v=100(\ln 2)^3$.

$$100[\ln(0+k)]^3+C=100(\ln 2)^3 \qquad *$$

$$100(\ln k)^3=100(\ln 2)^3$$

$$k=2 \qquad 1$$

*1M for both

12. (b) (ii) $\dfrac{dE}{dt}$

$$=\left[(2^M)(\ln 2)\left(\frac{dM}{dt}\right)\right]\left[\log\left(\frac{dv}{dt}\right)\right]^2+(2^M)\left\{\left[2\log\left(\frac{dv}{dt}\right)\right]\left(\frac{\frac{d^2v}{dt^2}}{\frac{dv}{dt}\ln 10}\right)\right\}$$

1M + 1A

$$=(2^M)(\ln 2)\left(\frac{dM}{dt}\right)\left[\log\left(\frac{dv}{dt}\right)\right]^2+\left(\frac{2^{M+1}}{\ln 10}\right)\left[\log\left(\frac{dv}{dt}\right)\right]\left(\frac{\frac{d^2v}{dt^2}}{\frac{dv}{dt}}\right)$$

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{t=T}$$

$$=(2^{\log_2 e})(\ln 2)\left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{t=T}\right)\left[\log\left(\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=T}\right)\right]^2+\left(\frac{2^{\log_2 e+1}}{\ln 10}\right)\left[\log\left(\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=T}\right)\right]\left(\frac{\left.\frac{\mathrm{d}^2v}{\mathrm{d}t^2}\right|_{t=T}}{\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=T}}\right)$$

1M

$$=(e\ln 2)\left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{t=T}\right)\left[\log\left(\frac{300(2)^2}{e^2}\right)\right]^2$$

$$=(e\ln 2)\left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{t=T}\right)\left[\log\left(\frac{1200}{e^2}\right)\right]^2$$

$$0\le\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{t=T}\le\left[\log\left(\frac{1200}{e^2}\right)\right]^2$$ 1M

$$0\le(e\ln 2)\left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{t=T}\right)\left[\log\left(\frac{1200}{e^2}\right)\right]^2\le\left[\log\left(\frac{1200}{e^2}\right)\right]^2$$

$$0\le\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{t=T}\le\frac{1}{e\ln 2}$$

$$0\le\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{t=T}<1$$

$T$ s after the start of the experiment, the mass of the particle increases at a rate below 1 unit/s.

The claim is agreed. 1A f.t.

**End of Marking Scheme**