HKUGA College

MOCK EXAMINATION (2021/2022)

Mathematics Compulsory Part

Paper 1

Marking Scheme

TOTAL MARKS: 105

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Time allowed:	2 hours 15 minutes	Form:	6
Name:		Class (No.):	( )

Teacher: CC / HC / JY / MS / MY / SKC / WC

INSTRUCTIONS

1. This paper consists of 26 pages including this cover page. The words "End of Paper" should appear on the last page.

2. Do not open this exam paper until instructed to do so.

3. This paper consists of **THREE** sections, A(1), A(2) and B. Each section carries 35 marks.

4. Attempt **ALL** questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.

5. Graph paper and supplementary answer sheets will be supplied on request. Write your name, class and class number on each sheet.

6. Unless otherwise specified, all working must be clearly shown.

7. Unless otherwise specified, numerical answers should be either exact or correct to 3 significant figures.

8. The diagrams in this paper are not necessarily drawn to scale.

9. The use of an HKEAA-approved calculator is permitted.

SECTION A(1) (35 marks)

1. Simplify $(x^2y)(-3xy^{-2})^4$ and express your answer with positive indices. (3 marks)

$(x^2y)(-3xy^{-2})^4$

$= (x^2y)(81x^4y^{-8})$ 1M for $(a^h)^k = a^{hk}$ or $(ab)^l = a^lb^l$

$= 81x^6y^{-7}$ 1M for $c^pc^q = c^{p+q}$ or $d^{-r} = \frac{1}{d^r}$

$= \frac{81x^6}{y^7}$ 1A

2. Make x the subject of the formula $\frac{1}{x}+\frac{2}{y}=\frac{3}{z}$. (3 marks)

$\frac{1}{x}+\frac{2}{y}=\frac{3}{z}$

$\frac{1}{x}=\frac{3}{z}-\frac{2}{y}$ 1M

$\frac{1}{x}=\frac{3y-2z}{yz}$ 1M

$x=\frac{yz}{3y-2z}$ 1A

Alternatively,

$\frac{1}{x}+\frac{2}{y}=\frac{3}{z}$

$xyz\left(\frac{1}{x}+\frac{2}{y}\right)=xyz\left(\frac{3}{z}\right)$ 1M

$yz+2xz=3xy$

$yz=3xy-2xz$

$yz=x(3y-2z)$ 1M

$x=\frac{yz}{3y-2z}$ 1A

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3. Factorize

(a) $12x^2 - xy - 6y^2$,

(b) $9x + 6y - 12x^2 + xy + 6y^2$.

(3 marks)

(a) $12x^2 - xy - 6y^2$

$= (3x + 2y)(4x - 3y)$ 1A

(b) $9x + 6y - 12x^2 + xy + 6y^2$

$= 3(3x + 2y) - (3x + 2y)(4x - 3y)$ 1M for using (a)

$= (3x + 2y)(3 - 4x + 3y)$ 1A

4. (a) Solve the inequality $\frac{20 - 8x}{3} \le -2(x - 1)$.

(b) Find the number of integers satisfying both inequalities $\frac{20 - 8x}{3} \le -2(x - 1)$ and $30 - 3x > 0$.

(4 marks)

(a) $\frac{20 - 8x}{3} \le -2(x - 1)$

$20 - 8x \le -6x + 6$

$-8x + 6x \le 6 - 20$ 1M for putting x on one side

$-2x \le -14$

$x \ge 7$ 1A

(b) $30 - 3x > 0$

$x < 10$ 1A

By (a), we have $7 \le x < 10$.

Thus, the required number is 3. 1A

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5. The marked price of a wallet is 56.25% higher than the cost. The wallet is sold at a discount of 20% on its marked price. After selling the wallet, the profit is \$100. Find the cost of the wallet. (4 marks)

Let \$$c$ be the cost of the wallet.
The marked price of the wallet
= \$ (1.5625c) 1M
The selling price of the wallet
= \$ (c + 100) 1M for cost + 100
$1.5625c \times (1 - 20\%) = c + 100$ 1M for marked price × (1 – 20%)
$c = 400$
Thus, the cost of the wallet is \$400. 1A

6. In a park, the ratio of the number of adults to the number of children is 14 : 9. If 2 adults leave the park and 3 children enter the park, then the ratio of the number of adults to the number of children is 4 : 3. Find the original number of children in the park. (4 marks)

Let the original number of adults and children in the park be $14k$ and $9k$ respectively, where k is a positive constant. 1A (can be absorbed)

$$\frac{14k-2}{9k+3}=\frac{4}{3}$$ 1M + 1A

$42k - 6 = 36k + 12$
$k = 3$
Thus, the original number of children in the park is 27. 1A

<u>Alternative solution</u>
Let a and c be the original number of adults and children in the park respectively.

$$\begin{cases} \dfrac{a}{c}=\dfrac{14}{9} & ---(1) \\ \dfrac{a-2}{c+3}=\dfrac{4}{3} & ---(2) \end{cases}$$ 1A + 1A

(1): $a=\dfrac{14c}{9}$ $---(3)$

Sub (3) into (2),

$$\frac{\frac{14c}{9}-2}{c+3}=\frac{4}{3}$$ 1M for eliminating one unknown

$$9\left(\frac{14c}{9}-2\right)=12(c+3)$$

$14c-18=12c+36$
$c = 27$ 1A
Thus, the original number of children in the park is 27.

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7. In a polar coordinate system, O is the pole. The polar coordinates of the points A and B are $(6\ ,\ 35°)$ and $(b\ ,\ 125°)$ respectively. It is given that $AB = 10$.

(a) Find b.

(b) Find the perpendicular distance from O to AB.

(4 marks)

(a) $\angle AOB = 125° - 35°$

$= 90°$ 1A

$b^2 + 6^2 = 10^2$

$b = 8$ 1A

only 1A will be given if students do not obtain $\angle AOB = 90°$

(b) Let the perpendicular distance from O to AB be d.

$\frac{10d}{2} = \frac{6 \times 8}{2}$ 1M

$d = 4.8$

$\therefore$ The perpendicular distance from O to AB is 4.8. 1A

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8. The length of a piece of thin metal wire is measured as 8.2 m correct to the nearest 0.1 m.

(a) Is it possible that the actual length of this metal wire exceeds 826 cm? Explain your answer.

(b) Is it possible to cut this metal wire into 28 pieces of shorter metal rods, with each length measured as 30 cm correct to the nearest cm? Explain your answer.

(5 marks)

(a) The maximum absolute error = 0.05 m.
The actual length of this metal rod
$< 8.2 + 0.05$ — 1M for $8.2 + 0.05$
$= 8.25$ m
$= 825$ cm
< 826 cm
This, it is not possible that the actual length of this metal rod exceeds 826 cm. — 1A f.t.

> If the <u>actual length</u> of this metal rod <u>exceeds 826 cm</u>, then the <u>measured length</u> of this metal rod <u>correct to the nearest 0.1 m</u> will be <u>at least 8.3 m</u>. — 1M for $8.26 \text{ m} \approx 8.3 \text{ m}$
> This, it is not possible that the actual length of this metal rod exceeds 826 cm. — 1A f.t.

(b) Let n be the number of pieces of shorter metal rods.
$n < \dfrac{825}{30-0.5}$ — 1M for $\dfrac{(a)}{30-0.5}$ (accept $n \le \dfrac{826}{30-0.5}$)
$n < 27.96610169$ — 1A (accept $n < 28$)
Therefore, the greatest possible value of n is 27.
Thus, it is not possible to cut this metal rod in that way. — 1A f.t.

> The least possible length of 28 pieces of metal rods that are measured as 30 cm
> $= (28)(30 - 0.5)$ — 1M for $28(30 - 0.5)$
> $= 826$ cm — 1A
> $= 8.26$ m
> > 8.25 m — accept $826 > 825$
> Thus, it is not possible to cut this metal rod in that way. — 1A f.t.

> The actual length of each shorter metal rod
> $< \dfrac{825}{28}$ — 1M for $\dfrac{(a)}{28}$
> ≈ 29.46428571 cm — 1A
> < 29.5 cm
> Thus, it is not possible to cut this metal rod in that way. — 1A f.t.

-1A f.t. throughout Q8 if "maximum length" is used.

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9. In Figure 1, AB produced and CD produced meet at the point E. It is given that $\angle ACE = \angle DBE$.

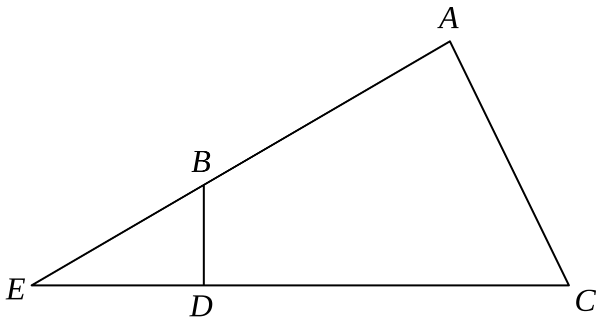

Figure 1

(a) Prove that $\Delta ACE \sim \Delta DBE$.

(b) It is given that $AC = 117$ cm, $EA = 240$ cm, $EC = 267$ cm and $BD = 39$ cm.

(i) Is ΔACE a right-angled triangle? Explain your answer.

(ii) Find the area of ΔBDE.

(5 marks)

(a)

$\angle ACE = \angle DBE$	(given)	
$\angle AEC = \angle DEB$	(common $\angle$)	
$\angle CAE = \angle BDE$	($\angle$ sum of Δ)	
$\therefore \Delta ACE \sim \Delta DBE$	(AA)	or (AAA)

Case 1:	Any correct proof with correct reasons.	2
Case 2:	Any correct proof without reasons.	1

(b) (i) $AC^2 + AE^2$

$= 117^2 + 240^2$

$= 71289$

EC^2

$= 267^2$

$= 71289$

$AC^2 + AE^2 = EC^2$

Thus, ΔACE is a right-angled triangle. 1A f.t.

(ii) $\frac{DE}{AE} = \frac{BD}{AC}$ 1M for corr. sides of ~Δs

$\frac{DE}{240} = \frac{39}{117}$

$DE = 80$ cm

Note that $\angle BDE = 90°$

The area of ΔBDE

$= \frac{80 \times 39}{2}$

$= 1560\text{ cm}^2$ 1A

pp-1 for without mentioning $\angle BDE = 90°$

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Page total

Alternative solution

The area of ΔBDE

$= \dfrac{(117)(240)}{2} \times \left(\dfrac{39}{117}\right)^2$ 1M for $\times \left(\dfrac{39}{117}\right)^2$

$= 1560 \text{ cm}^2$ 1A

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SECTION A(2) (35 marks)

10. The total cost for producing n cars is \$$S$, where n is a positive integer. S is partly constant and partly varies as n. When $n = 2$, $S = 280\ 000$. When $n = 6$, $S = 600\ 000$.

(a) Find the total cost for producing 20 cars. (4 marks)

(b) Peter claims that the total cost for producing $2n$ cars is less than twice the total cost for producing n cars for some positive integer n. Do you agree? Explain your answer. (2 marks)

(a) Let $S = a + bn$, where a and b are non-zero constants. 1A

When $n = 2$, $S = 280\ 000$, We have

$280\ 000 = a + 2b$ 1M for either

When $n = 6$, $S = 600\ 000$. We have

$600\ 000 = a + 6b$

By solving, we have $a = 120\ 000$ and $b = 80\ 000$. 1A for both

The total cost for producing 20 cars

$= 120\ 000 + 80\ 000(20)$

$= \$1\ 720\ 000$ 1A

(b) Total cost for producing $2n$ cars – 2 × total cost for producing n cars

$= 120\ 000 + 80\ 000(2n) - 2(120\ 000 + 80\ 000n)$ 1M for using (a)

$= -\$120\ 000$

$< \$0$

Thus, the claim is agreed. 1A f.t.

Alternative solution:

Total cost for $2n$ cars

$= 120\ 000 + 80\ 000(2n)$ 1M for using (a)

$= \$\ (120\ 000 + 160\ 000n)$

Total cost for n cars

$= \$(120\ 000 + 80\ 000n)$

2 × total cost for producing n cars

$= 2 \times (120\ 000 + 80\ 000n)$

$= \$\ (240\ 000 + 160\ 000n)$

$>$ Total cost for $2n$ cars

Thus, the claim is agreed. 1A f.t.

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11. The following table shows the distribution of the number of mock paper(s) done by a group of students in a certain week.

Number of mock paper(s)	1	2	3	4	5
Number of students	a	20	9	8	2

It is given that the mean of the above distribution is 2.75.

(a) Find a. (2 marks)

(b) Find the median, the inter-quartile range and the standard deviation of the above distribution. (3 marks)

(c) It is found that the number of mock paper(s) done by one of the students is wrongly recorded. After making the correction, the range of the distribution remains the same. Find the maximum and minimum possible values of the mean after making correction. (2 marks)

(a) $\frac{a+2(20)+3(9)+4(8)+5(2)}{a+20+9+8+2}=2.75$ 1M

$a = 1$ 1A

(b) Median = 2 1A

Interquartile range = 3.5 – 2

= 1.5 1A

Standard deviation = 0.968245836

= 0.968 1A (r.t. 0.968)

(c) Mean is maximum when 1 is changed to 6.

Maximum value of mean = 2.875 1A

Mean is minimum when 5 is changed to 1.

Minimum value of mean = 2.65 1A

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12. A glass container is in a shape of a vertically inverted right circular cone of base radius 48 cm and height 36 cm. Initially, the container is empty. Suppose the water is being added at a constant rate of $x\pi$ cm^3/s . The container will be full after 64 minutes.

(a) Find x. (2 marks)

(b) Find the wet surface area of the container when the water has been added to the container for 27 minutes. (4 marks)

(a) The capacity of the container

$= \frac{1}{3}\pi(48)^2(36)$ 1M for $\frac{1}{3}\pi r^2 h$

$= 27648\pi$ cm^3

$x\pi \times 64 \times 60 = 27648\pi$

$x = 7.2$ 1A

(b) The slant height

$= \sqrt{48^2 + 36^2}$ 1M for slant height

$= 60$ cm

The area of the lateral face of the container

$= \pi(48)(60)$ 1M for πrl

$= 2880\pi$ cm^2

Water inside the container after 27 minutes

$= 7.2\pi \times 27 \times 60$

$= 11664\pi$ cm^3

The area of the wet surface

$= (2880\pi) \times \left(\sqrt[3]{\frac{11664\pi}{27648\pi}}\right)^2$ 1M for area ratio $\left(\frac{3}{4}\right)^2$ or length ratio $\frac{3}{4}$

$= 1620\pi$ cm^2 1A (r.t. 5090 cm^2)

Page total

13. The equation of the circle C is $4x^2+4y^2-40x+96y-6549=0$. Let G be the centre of C. Denote the origin by O.

(a) Find OG. (2 marks)

(b) Does O lie inside C? Explain your answer. (1 mark)

(c) Let P be a moving point in the rectangular coordinate plane such that $OP = GP$. Denote the locus of P by Γ. Suppose that Γ cuts C at the points A and B. Find the area of the quadrilateral $OAGB$. (4 marks)

(a) $4x^2+4y^2-40x+96y-6549=0$

$x^2+y^2-10x+24y-\dfrac{6549}{4}=0$

Coordinates of G

$=\left(-\dfrac{-10}{2},-\dfrac{24}{2}\right)$

$=(5\ ,-12)$ 1A

OG

$=\sqrt{(5-0)^2+(-12-0)^2}$

$=13$ 1A

(b) Radius of C

$=\sqrt{\left(\dfrac{-10}{2}\right)^2+\left(\dfrac{24}{2}\right)^2-\left(-\dfrac{6549}{4}\right)}$

$=\dfrac{85}{2}$

>13

Thus, O lie inside C. 1A f.t.

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(c) By (b), we have $GA = \frac{85}{2}$.

Let M be the mid-point of AB.

Note that Γ is the perpendicular bisector of OG.

Since M lies on Γ, M is the mid-point of OG.

GM

$= \frac{1}{2}OG$

$= \frac{1}{2}(13)$ 1M for using (a)

$= \frac{13}{2}$

Also note that $\angle GMB = 90°$

MB

$= \sqrt{GB^2 - GM^2}$ 1M for Pyth. Theorem

$= \sqrt{\left(\frac{85}{2}\right)^2 - \left(\frac{13}{2}\right)^2}$

$= 42$

Since both A and B lie on Γ, we have $OA = GA$ and $OB = GB$.

Further note that $GA = GB$.

So, we have $OA = GA = GB = OB$.

Hence, the quadrilateral $OAGB$ is a rhombus.

The area of the quadrilateral $OAGB$

$= 4\left(\frac{1}{2}(GM)(MB)\right)$ 1M

$= 4\left(\frac{1}{2}\left(\frac{13}{2}\right)(42)\right)$

$= 546$ 1A

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14. Let $f(x)=6x^3-11x^2-15x-37$. When $f(x)$ is divided by $3x^2+ax-5$, the quotient and the remainder are $bx-3$ and $cx+d$ respectively, where a , b , c and d are constants.

(a) Find a and b.

(4 marks)

(b) Let g(x) be a polynomial with degree greater than 2. When g(x) is divided by $3x^2+ax-5$, the remainder is $cx+d$.

(i) Prove that f(x) – g(x) is divisible by $3x^2+ax-5$.

(ii) Edan claims that all the roots of the equation f(x) – g(x) = 0 are rational. Do you agree? Explain your answer.

(5 marks)

(a) Note that $f(x)=(3x^2+ax-5)(bx-3)+cx+d$ 1M

Hence, we have $f(x)=3bx^3+(ab-9)x^2+(-3a-5b+c)x+15+d$.

Also note that $f(x)=6x^3-11x^2-15x-37$.

By considering the coefficient of x^3, we have

$3b=6$ 1M for either

$b=2$ 1A

By considering the coefficient of x^2, we have

$2a-9=-11$

$a=-1$ 1A

(b) (i) Let $g(x)=Q(x)(3x^2+ax-5)+cx+d$, where $Q(x)$ is a polynomial.

f(x) – g(x)

$=(3x^2+ax-5)(bx-3)+cx+d-\left[Q(x)(3x^2+ax-5)+cx+d\right]$

$=(3x^2+ax-5)(bx-3)-Q(x)(3x^2+ax-5)$ 1M for eliminating $cx+d$

$=(3x^2+ax-5)(bx-3-Q(x))$

Thus, f(x) – g(x) is divisible by $3x^2+ax+5$. 1

(ii) f(x) – g(x) = 0

$(3x^2-x-5)(2x-3-Q(x))=0$ 1M for using (a) and (b)(i)

$3x^2-x-5=0$ or $2x-3-Q(x)=0$

For $3x^2-x-5=0$,

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(3)(-5)}}{2(3)}$$ 1M

$$=\frac{1\pm\sqrt{61}}{6}$$

Note that $\frac{1+\sqrt{61}}{6}$ and $\frac{1-\sqrt{61}}{6}$ are roots of the above equation.

So, not all the roots of the equation f(x) – g(x) = 0 are rational.

Thus, the claim is disagreed. 1A f.t.

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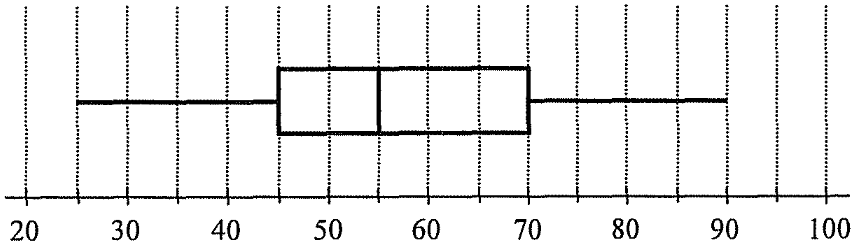
SECTION B (35 marks)

15. The box-and-whisker diagram below shows the distribution of the scores (in marks) of the students of a class in a test. Betty gets the lowest score while Ken gets 73 marks in the test. The standard scores of Betty and Ken in the test are –3 and 1 respectively.



(a) Find the mean of the distribution. (2 marks)

(b) Betty claims that the standard scores of at least half of the students in the test are positive. Do you agree? Explain your answer. (2 marks)

(a) Note that the lowest score of the distribution is 25 marks.
Let m marks and d marks be the mean and the standard deviation of the distribution respectively.

$$\begin{cases} 25 = m - 3d \\ 73 = m + d \end{cases}$$ 1M

Solving, we have $m = 61$ 1A

Thus, the mean of the distribution is 61 marks.

(b) Median = 55 marks
Standard score of the students who obtained 55 marks

$= \dfrac{55-61}{12}$ 1M for $\dfrac{median - \bar{x}}{\sigma}$

$= -0.5$

< 0

The standard scores of at least half of the students are negative.
Therefore, the statement "the standard scores of at least half of the students in the test are positive" is not always true.
Thus, the claim is disagreed. 1A f.t.

(b) Note that if the <u>test</u> score of a students is <u>higher than the mean</u>, then the <u>standard score of the students is positive</u>. 1M for either
Also note that the <u>median is 55 marks</u> and the <u>mean is 61 marks</u>.
So the <u>median is less than the mean</u>.
The test scores of at least half of the students are lower than the mean. i.e., the test scores of at most half of the students are higher than the mean.
Therefore, the statement "the standard scores of at least half of the students in the test are positive" is not always true.
Thus, the claim is disagreed. 1A f.t.

Page total

Remark for Q15b:

When the number of students is odd, then there exist at least one student who has a result of the median (55 marks). For this case, less than 50% of the students have results greater than the median.

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16. There are 7 boys and 9 girls in a class. 5 students are randomly selected from the class to form a committee.

(a) How many different committee can be formed? (2 marks)

(b) What is the probability that the committee formed only have boys? (2 marks)

(b) What is the probability that the committee formed have members of both genders? (2 marks)

(a) The required number

$= C_5^{16}$ 1M

$= 4368$ 1A

(b) The required probability

$= \dfrac{C_5^7}{C_5^{16}}$ 1M for $\dfrac{C_5^7}{(a)}$

$= \dfrac{1}{208}$ 1A

(c) The required probability

$= 1 - \dfrac{1}{208} - \dfrac{C_5^9}{C_5^{16}}$ 1M for $1-(b)-\dfrac{C_5^9}{(a)}$

$= \dfrac{201}{208}$ 1A

The required probability

$= \dfrac{C_1^7 C_4^9 + C_2^7 C_3^9 + C_3^7 C_2^9 + C_4^7 C_1^9}{C_5^{16}}$ 1M

$= \dfrac{201}{208}$ 1A

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17. Let α and β be real numbers such that $\begin{cases} \beta = 3\alpha - 4 \\ \beta = \alpha^2 - 5\alpha + 12 \end{cases}$.

(a) Find α and β. (2 marks)

(b) The 1st term and the 2nd term of an arithmetic sequence are log α and log β respectively. Find the least value of n such that the sum of the first n terms of the sequence is greater than 2022. (4 marks)

(a) Putting $\beta = 3\alpha - 4$ in $\beta = \alpha^2 - 5\alpha + 12$, we have

$3\alpha - 4 = \alpha^2 - 5\alpha + 12$ 1M

$\alpha^2 - 8\alpha + 16 = 0$

Solving, we have $\alpha = 4$ and $\beta = 8$. 1A for both

(b) Let T(n) be the nth term of the arithmetic sequence.

T(1) = log4 = $\log 2^2$ = 2 log2 1M for either

T(2) = log8 = $\log 2^3$ = 3 log2

The common difference of the sequence is log2.

T(1) + T(2) + T(3) + … + T(n) > 2022

2 log2 + 3 log2 + 4 log2 + … + (n + 1) log2 > 2022

$\frac{n}{2}\left(2\log 2 + (n+1)\log 2\right) > 2022$ 1M for $sum = \frac{(a+l)n}{2}$

$(\log 2)n^2 + (3\log 2)n - 4044 > 0$

$n < -117.4143098$ or $n > 114.4143098$ 1M

Thus, the least value of n is 115. 1A

Page total

18. Figure 2 shows a geometric model $ABCD$ in the form of tetrahedron where ΔBCD lies on a horizontal plane. It is given that $AB = 15$ cm, $AC = 19$ cm, $\angle ABD = 90°$, $\angle BAC = 49°$, $\angle BCD = 51°$ and $\angle BDC = 31°$.

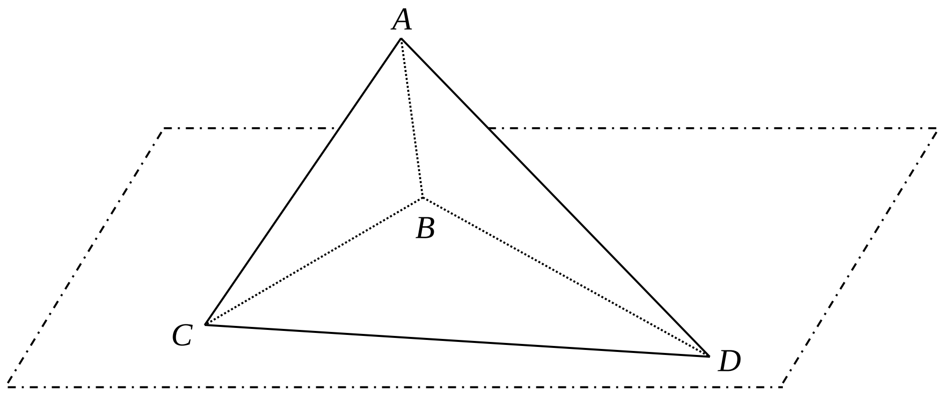

Figure 2

(a) Find BC and AD. (5 marks)

(b) Let P be a movable point on CD. Find the greatest possible inclination of AP. (4 marks)

(a) In ΔABC,

$BC^2 = AC^2 + AB^2 - 2(AC)(AB)\cos\angle BAC$ 1M

$= (19)^2 + (15)^2 - 2(19)(15)\cos 49°$

≈ 212.0463535

$BC \approx 14.56181148$ cm

$BC \approx 14.6$ cm 1A r.t. 14.6 cm

In ΔBCD,

$\frac{BD}{\sin\angle BCD} = \frac{BC}{\sin\angle BDC}$ 1M

$\frac{BD}{\sin 51°} \approx \frac{14.56181148}{\sin 31°}$

$BD \approx 21.97245899$ cm

$AD = \sqrt{AB^2 + BD^2}$ 1M

$\approx \sqrt{15^2 + 21.97245899^2}$

≈ 26.6043033 cm

≈ 26.6 cm 1A r.t. 26.6 cm

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(b) In ΔBCD,

$$\frac{CD}{\sin\angle CBD}=\frac{BC}{\sin\angle BDC}$$

$$\frac{CD}{\sin(180°-51°-31°)}\approx\frac{14.56181148}{\sin 31°}$$

$CD\approx 27.99811826$ cm

In ΔACD,

$$\cos\angle ACD=\frac{AC^2+CD^2-AD^2}{2(AC)(CD)}$$

$$\cos\angle ACD\approx\frac{19^2+27.99811826^2-27.15265252^2}{2(19)(27.99811826)}$$

$\angle ACD\approx 65.74231062°$ — 1M for both $\angle ACD$ and $\angle ACB$

In ΔABC,

$$\cos\angle ACB=\frac{AC^2+BC^2-AB^2}{2(AC)(AB)}$$

$$\cos\angle ACB\approx\frac{19^2+14.56181148^2-15^2}{2(19)(14.56181148)}$$

$\angle ACB\approx 51.02495766°$

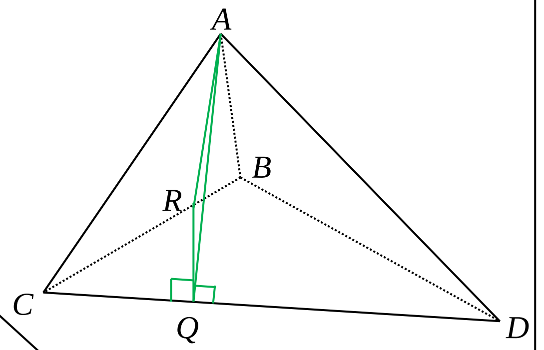
Let Q be the foot of perpendicular from A to CD.

Let R be a point on BC where $RQ\perp CD$.

The greatest possible inclination of AP will be obtained when P is at Q.

i.e.: The greatest possible inclination of $AP=\angle AQR$

1M for identifying the maximum size of the inclination



In ΔACQ,

$AQ=AC\sin\angle ACQ$

$\approx 19\sin 65.74231062°$

≈ 17.32243138 cm

$CQ=AC\cos\angle ACQ$

$\approx 19\cos 65.74231062°$

≈ 7.805983026 cm

In ΔCQR,

$QR=CQ\tan\angle RCQ$ — 1M for either AQ, QR or AR

$\approx 7.805983026\tan 51°$

≈ 9.639586243 cm

$$CR=\frac{CQ}{\cos\angle RCQ}$$

$$\approx\frac{7.805983026}{\cos 51°}$$

≈ 12.40382981 cm

In ΔACR,

$AR^2=AC^2+CR^2-2(AC)(CR)\cos\angle ACR$

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$\approx 19^2 + 12.40382981^2 - 2(19)(12.40382981)\cos 51.02495766°$

≈ 218.3872268

$AR \approx 14.7779304$ cm

In ΔAQR,

$$\cos \angle AQR = \frac{AQ^2 + QR^2 - AR^2}{2(AQ)(QR)}$$

$$\cos \angle AQR \approx \frac{(17.32243138)^2 + (9.639586243)^2 - (14.7779304)^2}{2(17.32243138)(9.639586243)}$$

$\angle AQR \approx 58.47861145°$

The greatest possible inclination of *AP* is 58.5° 1A r.t. 58.5°

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19. Let $\mathrm{f}(x) = 2x^2 - 4(k-1)x + 4k^2 - 4k - 8$, where k is a positive constant and $k \neq 1$. P is the vertex of the graph of $y = \mathrm{f}(x)$.

(a) Using the method of completing the square, express the coordinates of P in terms of k.

(2 marks)

(b) The graph of $y = \mathrm{g}(x)$ is obtained by reflecting the graph of $y = \mathrm{f}(x)$ in x–axis and then translating the resulting graphs upwards by 8 units. Let Q be the vertex of the graph of $y = \mathrm{g}(x)$. Denote the origin by O.

(i) Write down, in terms of k, the coordinates of Q.

(ii) Is it possible that the circumcentre of ΔOPQ lies on the x–axis? Explain your answer.

(iii) The coordinates of the point R are (–5 , 4). It is given that the graph of $y = \mathrm{f}(x)$ passes through O. Are P, Q, O and R concyclic? Explain your answer.

(8 marks)

(a) $\mathrm{f}(x) = 2x^2 - 4(k-1)x + 4k^2 - 4k - 8$

$= 2\left[x^2 - 2(k-1)x\right] + 4k^2 - 4k + 8$

$= 2\left[x^2 - 2(k-1)x + (k-1)^2 - (k-1)^2\right] + 4k^2 - 4k - 8$ 1M

$= 2\left(x - (k-1)\right)^2 + 2k^2 - 10$

The coordinates of P are $(k - 1, 2k^2 - 10)$ 1A

(b) (i) Coordinates of $Q = (k - 1, -(2k^2 - 10) + 8)$

$= (k - 1, 18 - 2k^2)$ 1A

(ii) Note that the x-coordinate of P = x-coordinate of Q.

PQ is a vertical line.

So, the perpendicular bisector of PQ is a horizontal line. 1M for either

The y-coordinate of the circumcentre of ΔOPQ

$= \dfrac{(2k^2 - 10) + (18 - 2k^2)}{2}$ 1M

$= 4$

$\neq 0$

Therefore, the circumcentre of ΔOPQ does not lie on the x-axis.

Thus, it is not possible. 1A f.t.

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(iii) $4k^2 - 4k - 8 = 0$

$k = 2$ or $k = -1$ (rejected) — 1A

Coordinates of $P = (1, -2)$

Coordinates of $Q = (1, 10)$

Slope of PR × Slope of QR

$= \dfrac{-2-4}{1-(-5)} \times \dfrac{10-4}{1-(-5)}$ — 1M for considering $m_{PR} \times m_{QR}$

$= -1$

$\therefore \angle PRQ = 90°$

Slope of OP × Slope of OR

$= \dfrac{-2-0}{1-0} \times \dfrac{10-0}{1-0}$ — 1M for considering $m_{OP} \times m_{OR}$

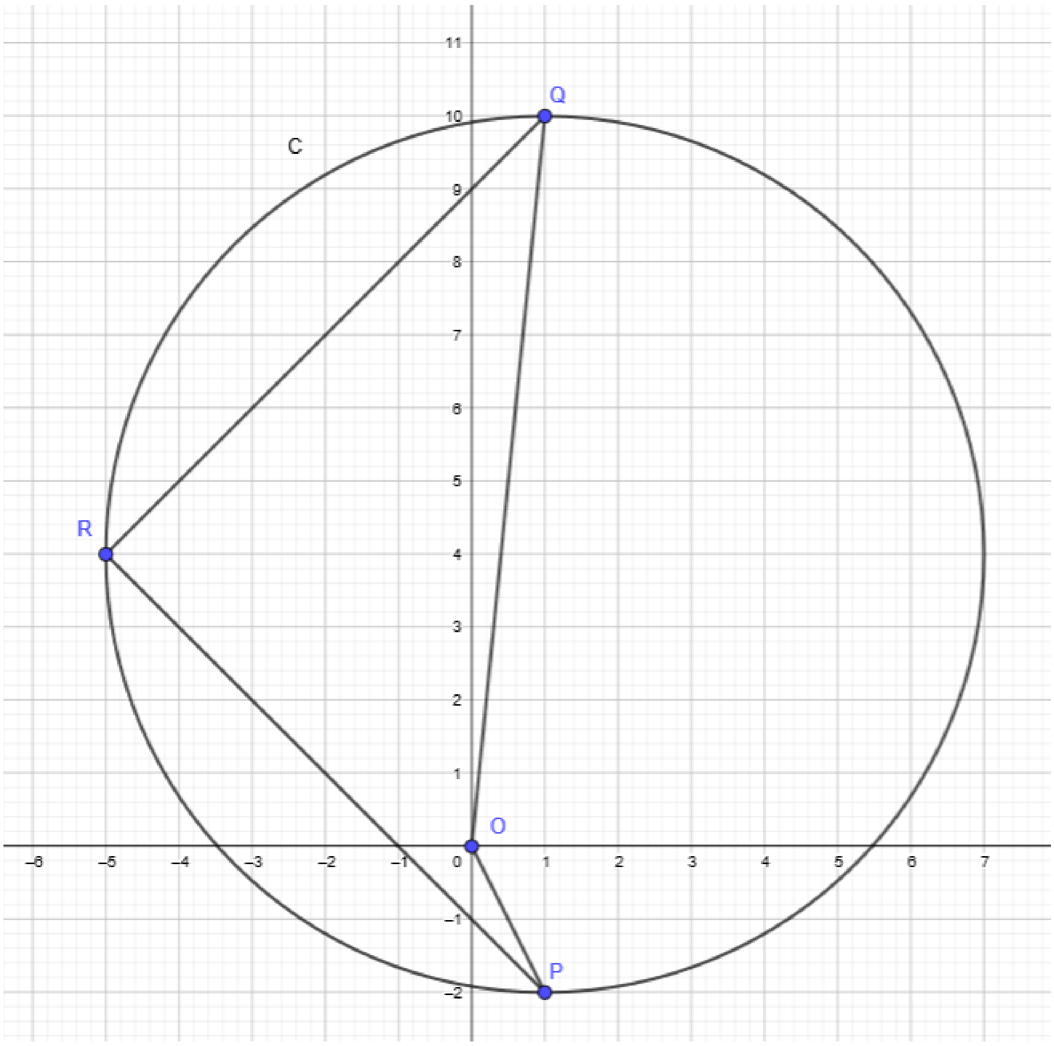
$= -20$

$\neq -1$

$\therefore \angle POQ \neq 90°$

$\therefore \angle PRQ \neq \angle POQ$

Thus, P, Q, O and R are not concyclic. — 1A f.t.



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(iii) $4k^2 - 4k - 8 = 0$

$k = 2$ or $k = -1$ (rejected) 1A

Coordinates of $P = (1, -2)$

Coordinates of $Q = (1, 10)$

Denote the mid-point of PQ be M

Coordinates of $M = \left(1, \frac{-2+10}{2}\right)$

$= (1, 4)$

$PM = QM = 10 - 4 = 6$

$RM = 1 - (-5)$

$= 6$

Note that $RM = PM = QM = 6$.

Thus, M is the centre of the circle passing through R, P and Q with radius 6 units. 1M

OM

$= \sqrt{(1-0)^2 + (4-0)^2}$

$= \sqrt{17}$

< 6 1M for comparing OM with the radius of circle

Therefore, O lies inside the above-mentioned circle.

Thus, P, Q, O and R are not concyclic. 1A f.t.

Page total

(iii) $4k^2 - 4k - 8 = 0$

$k = 2$ or $k = -1$ (rejected) 1A

Coordinates of $P = (1, -2)$

Coordinates of $Q = (1, 10)$

Let $x^2 + y^2 + Dx + Ey + F = 0$ be the equation of the circle passing through P, Q and O.

Sub $(0, 0)$, we have $F = 0$.

Sub $(1, -2)$, we have $1 + 4 + D - 2E = 0$ 1M for either

Sub $(1, 10)$, we have $1 + 100 + D + 10E = 0$

By solving, we have $D = -21$, $E = -8$

The equation of the circle is $x^2 + y^2 - 21x - 8y = 0$

Sub $(-5, 4)$ into $x^2 + y^2 - 21x - 8y = 0$,

L.H.S. $= (-5)^2 + 4^2 - 21(-5) - 8(4)$ 1M

$= 114$

$\neq 0$

$\therefore$ The circle which passes through P, Q and O does not pass through the point R.

Thus, P, Q, O and R are not concyclic. 1A f.t.

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(iii) $4k^2 - 4k - 8 = 0$

$k = 2$ or $k = -1$ (rejected) 1A

Coordinates of $P = (1, -2)$

Coordinates of $Q = (1, 10)$

$OR = \sqrt{(-5-0)^2 + (4-0)^2} = \sqrt{41}$

$OP = \sqrt{(1-0)^2 + (-2-0)^2} = \sqrt{5}$

$RP = \sqrt{(-5-1)^2 + (4-(-2))^2} = 6\sqrt{2}$

$OQ = \sqrt{(1-0)^2 + (10-0)^2} = \sqrt{101}$

$RQ = \sqrt{(-5-1)^2 + (4-10)^2} = 6\sqrt{2}$

$$\cos\angle RQO = \frac{\left(6\sqrt{2}\right)^2 + \left(\sqrt{101}\right)^2 - \left(\sqrt{41}\right)^2}{2\left(6\sqrt{2}\right)\left(\sqrt{101}\right)}$$

$\angle RQO \approx 39.28940686°$

$$\cos\angle RPO = \frac{\left(6\sqrt{2}\right)^2 + \left(\sqrt{5}\right)^2 - \left(\sqrt{41}\right)^2}{2\left(6\sqrt{2}\right)\left(\sqrt{5}\right)}$$

1M for either

$\angle RPO \approx 18.43494882°$

$\angle RQO + \angle RPO$

$\approx 18.43494882° + 39.28940686°$

$\approx 57.72435569°$

$\neq 180°$

1M for considering a pair of opp. $\angle$s

Also note that $\angle RQO \neq \angle RPO$.

$\therefore$ The circle which passes through P, Q and O does not pass through the point R.

Thus, P, Q, O and R are not concyclic. 1A f.t.

END OF PAPER

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