Methodist College

First Mock Examination, 2022 – 2023

F.6 Mathematics

Paper 1 Marking Scheme

Name: ________________________ Marks: ______________ / 105

Class: __________ Class No.: __________ Date: 20 October 2022

Time: 8:30 – 10:45 (2 hr. 15 min.)

INSTRUCTIONS:

1. This paper consists of THREE sections, A(1), A(2) and B.
2. Attempt ALL questions in this paper.
   Write your answers in the spaces provided in this Question-Answer Book.
3. Unless otherwise specified, all working must be clearly shown.
4. Unless otherwise specified, numerical answers should be either exact or correct to 3 significant figures.
5. The diagrams in this paper are not necessarily drawn to scale.
6. Students are allowed to use the calculator.

SECTION A(1) (35 marks)

1. Simplify $\frac{a^6b^{-4}}{(a^{-5}b^2)^3}$ and express your answer with positive indices. [index] (3 marks)

Solution.

$\frac{a^6b^{-4}}{(a^{-5}b^2)^3}$

$= \frac{a^6b^{-4}}{a^{-15}b^6}$ 1A+1A

$= \frac{a^{21}}{b^{10}}$ 1A

3

2. Let x and y be two numbers. The sum of x and y is 567 while the product of 8 and x is y. Find y. [equations] (3 marks)

Solution.

$\begin{cases} x + y = 567 \ldots(1) \\ 8x = y \ldots(2) \end{cases}$ 1A

Sub. (2) into (1),

$x + 8y = 567$ 1M

$x = 63$

$y = 504$ 1A

3

3. Simplify $\frac{4}{m+8} - \frac{3}{5m+6}$. [polynomials]

(3 marks)

Solution.

$$\frac{4}{m+8} - \frac{3}{5m+6}$$

$$= \frac{4(5m+6)-3(m+8)}{(m+8)(5m+6)}$$ 1A

$$= \frac{20m+24-3m-24}{(m+8)(5m+6)}$$ 1A

$$= \frac{17m}{(m+8)(5m+6)}$$ 1A

3

4. Factorize [factorization]

(a) $p^2 - 4pq + 4q^2$,

(b) $(2p + 1)^2 - p^2 + 4pq - 4q^2$.

(4 marks)

Solution.

(a) $p^2 - 4pq + 4q^2$

$= (p - 2q)^2$ 1A

(b) $(2p + 1)^2 - p^2 + 4pq - 4q^2$

$= (2p + 1)^2 - (p^2 - 4pq + 4q^2)$

$= (2p + 1)^2 - (p - 2q)^2$ 1M

$= [(2p + 1) - (p - 2q)][(2p + 1) + (p - 2q)]$ 1M

$= (2p + 1 - p + 2q)(2p + 1 + p - 2q)$

$= (p + 2q + 1)(3p - 2q + 1)$ 1A

4

5. A clock is sold at a discount of 35% on its marked price. After selling the clock, the loss is $110 and the percentage loss is 22%. Find the marked price of the clock. [percentage] (4 marks)

Solution.

$C \times 22\% = 110$ 1M

$C = 500$

$S = 500 - 110 = 390$ 1A

$390 = M(1 - 35\%)$ 1M

$M = \$600$ 1A

4

6. Consider the compound inequality

$-3(2x - 1) > x + 10$ or $3x + 9 \leq 0$ .... (*).

(a) Solve (*).

(b) Write down the greatest integer satisfying (*).

[inequalities] (4 marks)

Solution.

(a)	$-3(2x - 1) > x + 10$	or	$3x + 9 \leq 0$	
	$-6x + 3 > x + 10$	or	$3x \leq -9$	
	$-7x > 7$	or	$x \leq -3$	
	$x < -1$	or	$x \leq -3$	1A+1A
$\therefore$	$x < -1$			1A
(b)	-2			1A

4

7. The coordinates of the points M and N are (–11, 6) and (–2, –5) respectively. M is rotated anticlockwise about O through 90° to M', where O is the origin. N' is the reflection image of N with respect to the y-axis.[polar coordinates, rotation, reflection and symmetry]

(a) Write down the coordinates of M' and N'.

(b) Find the slope of $M'N'$.

(4 marks)

Solution.

(a) $M' = (-6, -11)$; $N' = (2, -5)$ 1A+1A

(b) Slope of $M'N'$

$= \dfrac{-11-(-5)}{-6-2}$ 1M

$= \dfrac{3}{4}$ 1A

4

8. In Figure 1, $ABCDE$ is a pentagon. $AC \parallel ED$ and $AD \parallel BC$. It is given that $\angle ABC = \angle AED$ and $AB = AE$. [plane geometry]

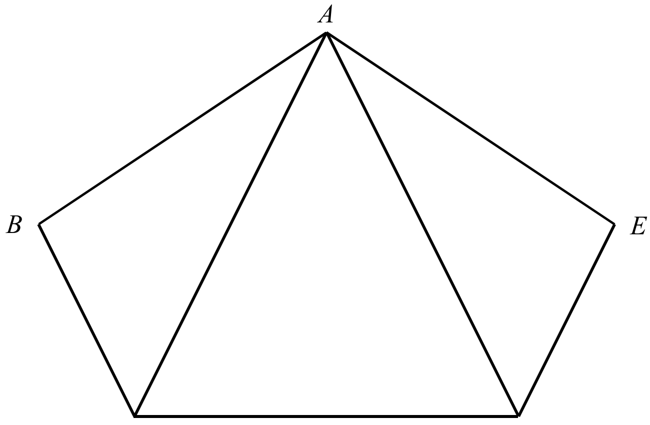

Figure 1

(a) Prove that $\Delta ABC \cong \Delta AED$.

(b) If $\angle ABC = 78^\circ$ and $\angle DAE = 39^\circ$, find $\angle ACD$.

(5 marks)

Solution.

(a)	$AB = AE$	(given)	
	$\angle ABC = \angle AED$	(given)	
	$\angle ACB = \angle CAD$	(alt. $\angle$s, $AC \parallel ED$)	
	$\angle ADE = \angle CAD$	(alt. $\angle$s, $AD \parallel BC$)	1A (either one)
	$\therefore \angle ACB = \angle ADE$		
	$\Delta ABC \cong \Delta AED$	(A.A.S.)	1
(b)	$\angle BAC = \angle DAE = 39^\circ$	(cor. $\angle$s, $\cong \Delta$)	
	$\angle ACB = 180^\circ - \angle ABC - \angle BAC$	($\angle$ sum of Δ)	
	$\angle ACB = 180^\circ - 78^\circ - 39^\circ = 63^\circ$		1A
	$AC = AD$	(cor. sides, $\cong \Delta$)	
	$\angle ACD = \angle ADC$	(base $\angle$s, isos. Δ)	1M
	$\angle ACB + \angle ACD + \angle ADC = 180^\circ$	(int. $\angle$s, $AD \parallel BC$)	
	$63^\circ + 2\angle ACD = 180^\circ$		
	$\angle ACD = 58.5^\circ$		1A

5

9. The frequency distribution table and the cumulative frequency distribution table below show the distribution of the time taken to complete a homework by a group of students. [statistics]

Time taken (minutes)	Frequency
40 – 44	a
45 – 49	7
50 – 54	b
55 – 59	5

Time taken less than (minutes)	Cumulative frequency
44.5	5
49.5	x
54.5	y
59.5	30

(a) Write down the value of b.

(b) Find the mean of the distribution.

(c) Find the probability that the time taken to complete the homework by a randomly selected student from the group is not less than 49.5 minutes.

(5 marks)

Solution.

(a) $b = 13$ 1A

(b) mean

$= \frac{5(42)+7(47)+13(52)+5(57)}{30}$ 1M

$= 50$ 1A

(c) Probability

$= \frac{30-12}{30}$ 1M

$= \frac{3}{5}$ 1A

5

SECTION A(2) (35 marks)

10. It is given that $f(x)$ is partly constant and partly varies as $(x-2)^2$. Suppose that $f(6) = -36$ and $f(-3) = -63$. [variation]

(a) Find $f(x)$. (3 marks)

(b) Write down the x-intercept(s) of the graph of $y = 3f(x)$. (1 mark)

(c) Let k be a real constant. Find the range of values of k such that the equation $f(x) = k$ has no real roots. (2 marks)

Solution.

(a) Let $f(x) = a(x-2)^2 + b$, where a and b are non-zero constant.

$$\begin{cases} a(6-2)^2 + b = -36 & (1) \\ a(-3-2)^2 + b = -63 & (2) \end{cases}$$ 1M (either)

(1) – (2),

$-9a = 27$

$a = -3$

$16(-3) + b = -36$

$b = 12$ 1A (either one)

$f(x) = -3(x-2)^2 + 12$ 1A

(b) 0, 4 1A

(c) $f(x) = -3(x-2)^2 + 12$

Maximum value of $f(x)$ is 12. 1M

When $f(x) = k$ has no real root

$k > 12$ 1A

6

11. The stem-and-leaf diagram below shows the distribution of the weights (kg) of the members of a drama club. [statistics]

Stem (tens)	Leaf (units)
5	4 6 7
6	0 a a 7 7 8
7	b b 4 4 5 5 5 5 6 8
8	3

The inter-quartile range and the median of the distribution are 14 kg and 71 kg respectively.

(a) Find a and b. (3 marks)

(b) A new member now joins the drama club.

(i) Is there any change in the median of the distribution due to the joining of the new member?

(ii) If the range of the distribution is increased by 1, find the greatest possible standard deviation of the distribution.

(4 marks)

Solution.

(a) $75 - (60 + a) = 14$ 1A

$a = 1$ 1A

$70 + b = 71$

$b = 1$ 1A

(b) (i) No 1A

(ii) Range increased by 1. Weight of new member must be 53 kg or 84 kg. 1M

Standard deviation

$= 8.52$ or 8.45 1A (either one)

Greatest possible standard deviation is 8.52. 1A

7

12. The equation of the circle C is $x^2 + y^2 - 30x - 40y + 369 = 0$. Denote the centre of C by G. The coordinates of the point H are (45, 36). [equations of circles]

(a) Find the distance between G and H. (3 marks)

(b) Let P be a moving point on C. When the $\angle GHP$ is the greatest,

(i) describe the geometric relationship between HP and GP;

(ii) find the area of ΔGHP.

(4 marks)

Solution.

(a) Centre = (15, 20) 1A

$GH^2 = (45 - 15)^2 + (36 - 20)^2$ 1M

$GH = 34$ 1A

(b) (i) HP is perpendicular to GP. 1A

(ii) GP = radius = $\sqrt{15^2 + 20^2 - 369} = 16$ 1A

$HP^2 + GP^2 = GH^2$

$HP = 30$ 1A

Area of ΔGHP

$= \dfrac{16 \times 30}{2}$

$= 240$ 1A

7

13. There are three solid metal spheres X, Y and Z. X is the smallest while Z is the largest. The ratio of the surface areas of X, Y and Z is $1 : 4 : 9$. The radius of X is 3 cm. [mensuration]

(a) Express, in terms of π, the volume of Z. (3 marks)

(b) These three spheres are melted and recast into two solid right circular cones. Denote these two circular cones by A and B. It is given that the height and the base radius of A are 12 cm and 6 cm respectively. A student finds that the base radius of B is 12 cm. The student claims that A and B are similar. Is the claim correct? Explain your answer. (4 marks)

Solution.

(a) Ratio of surface areas of X, Y and Z is $1 : 4 : 9$.

Ratio of radii of X, Y and Z is $1 : 2 : 3$.

Radius of X is 3 cm.

Radius of Z is 9 cm. 1A

Volume of Z

$= \frac{4}{3}\pi(9\text{ cm})^3$ 1M

$= 972\pi\text{ cm}^3$ 1A

(b) Radius of Y is 6 cm.

$\frac{1}{3}\pi(6)^2(12) + \frac{1}{3}\pi(12)^2 h = \frac{4}{3}\pi(3)^3 + \frac{4}{3}\pi(6)^3 + 972\pi$ 1M

$144\pi + 48\pi h = 36\pi + 288\pi + 972\pi$

$h = 24$ cm 1A

Ratio of radius of A to radius of B = 6 cm : 12 cm = 1 : 2

Ratio of height of A to height of B = 12 cm : 24 cm = 1 : 2

The two ratios are the same. 1M

Yes, the claim is agreed. 1A

7

14. Let $p(x) = 3x^3 + ax^2 + bx - 10$, where a and b are constants. When $p(x)$ is divided by $x^2 - 2x + 2$, the remainder is $-x - 2$. [polynomials]

(a) Find a and b. (3 marks)

(b) Is $x - 2$ a factor of $p(x)$? Explain your answer. (2 marks)

(c) Someone claims that the equation $p(x) = 0$ has two irrational roots. Do you agree? Explain your answer. (3 marks)

Solution.

(a) Let $3x^3 + ax^2 + bx - 10 = (mx + n)(x^2 - 2x + 2) - x - 2$ 1M

$3x^3 + ax^2 + bx - 10 = mx^3 + nx^2 - 2mx^2 - 2nx + 2mx + 2n - x - 2$

$3 = m$, $a = n - 2m$, $b = -2n + 2m - 1$, $-10 = 2n - 2$ 1A

$m = 3$, $n = -4$, $a = -10$, $b = 13$ 1A

(b) $f(2) = 3(2)^3 - 10(2)^2 + 13(2) - 10 = 0$ 1M

$(x - 2)$ is a factor of $p(x)$ 1A

(c) $p(x) = 0$

$(x - 2)(3x^2 - 4x + 5) = 0$ 1A

$x = 2$ or $3x^2 - 4x + 5 = 0$

$\Delta = (-4)^2 - 4(3)(5) = -44 < 0$ 1M

There are no real roots for the equation $3x^2 - 4x + 5 = 0$.

The claim is disagreed. 1A

8

SECTION B (35 marks)

15. There are 14 girls and 16 boys in a class. If 4 students are randomly selected from the class to form a debate team, find the probability that [counting and probabilities]

(a) there are 1 girl and 3 boys in the debate team; (2 marks)

(b) the number of girls is not less than the number of boys in the debate team. (2 marks)

Solution.

(a) $\dfrac{C_1^{14} \times C_3^{16}}{C_4^{14+16}}$ 1M

$= \dfrac{224}{783}$ 1A

(b) $1 - \dfrac{224}{783} - \dfrac{C_0^{14} \times C_4^{16}}{C_4^{14+16}}$ 1M

$= \dfrac{169}{261}$ 1A

4

16. Let $g(x) = 2x^2 + 12kx + 20k^2 + 8$, where k is a non-zero real constant. [quadratic functions and graphs]

(a) Using the method of completing the square, express, in terms of k, the coordinates of the vertex of the graph of $y = g(x)$. (2 marks)

(b) On the same rectangular coordinates system, denote the vertex of the graph of $y = g(x)$ and the vertex of the graph of $y = g(-x) + 3$ by A and B respectively. Let M be a point lying on AB such that the x-coordinate of M is k.

(i) Find the ratio $AM : BM$.

(ii) Find the y-coordinate of M. (3 marks)

Solution.

(a) $2x^2 + 12kx + 20k^2 + 8$

$= 2(x^2 + 6kx) + 20k^2 + 8$

$= 2(x^2 + 6kx + 9k^2 - 9k^2) + 20k^2 + 8$ 1M

$= 2[(x + 3k)^2 - 9k^2] + 20k^2 + 8$

$= 2(x + 3k)^2 + 2k^2 + 8$

Vertex $= (-3k, 2k^2 + 8)$ 1A

(b) $A = (-3k, 2k^2 + 8)$, $B = (3k, 2k^2 + 11)$ 1A

(i) $AM : BM = k - (-3k) : 3k - k = 2 : 1$ 1A

(ii) $y = \dfrac{(2k^2 + 8) + 2(2k^2 + 11)}{2 + 1} = 2k^2 + 10$ 1A

5

17. Let k be a real constant. The roots of the equation $x^2 - kx + 10 = 0$ are α and β. [sequences]

(a) Express $(\alpha - \beta)^2$ in terms of k. (3 marks)

(b) The 1st term, the 2nd term and the 3rd term of an arithmetic sequence are $(\alpha - \beta)^2$, k^2 and 121 respectively. Find the least value of n such that the sum of the first n terms of the sequence is greater than 2×10^6. (4 marks)

Solution.

(a) $(\alpha - \beta)^2$

$= (\alpha + \beta)^2 - 4\alpha\beta$ 1M

$= k^2 - 4(10)$ 1A

$= k^2 - 40$ 1A

(b) $d = k^2 - (\alpha - \beta)^2 = 40$ 1A

$a = 121 - 2(40) = 41$ 1A

Consider

$\frac{n}{2}[2(41) + (n-1)(40)] > 2 \times 10^6$ 1M

$41n + 20n(n-1) > 2 \times 10^6$

$20n^2 + 21n - 2 \times 10^6 > 0$

$n < -316.8$ or $n > 315.7$

The least value of n is 316. 1A

7

18. In Figure 2, the triangle paper card PQR is held such that PQ lies on the horizontal ground. It is given that $PQ = 30$ cm, $\angle PQR = 40°$ and $\angle QPR = 95°$. [application of trigonometry]

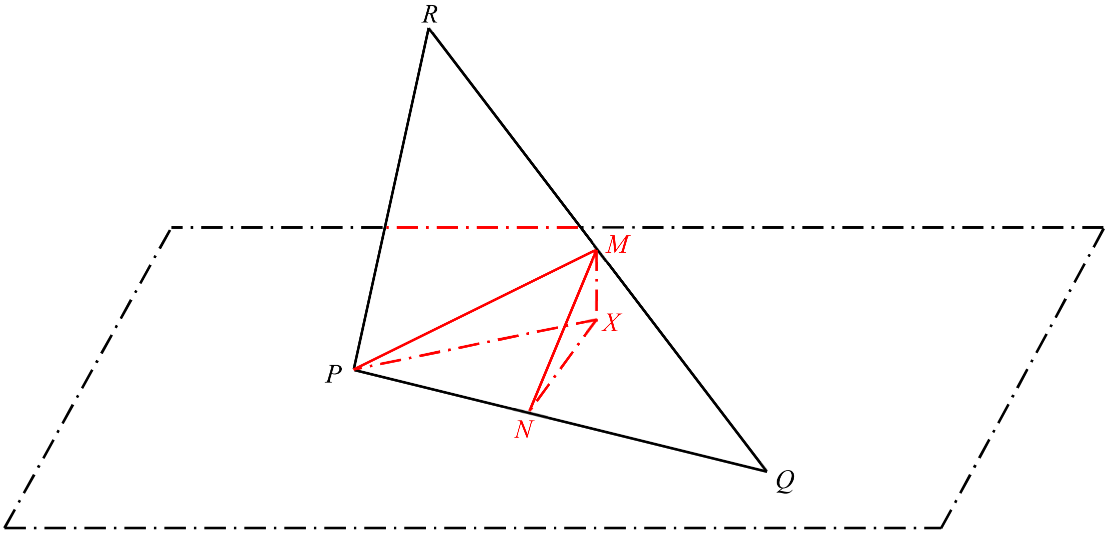

Figure 2

(a) Find the length of QR. (2 marks)

(b) Let M be the mid-point of QR. Find the length of PM. (2 marks)

(c) A craftsman finds that the angle between the plane PQR and the horizontal ground is 70º. The craftsman claims that the angle between PM and the horizontal ground exceeds 40º. Is the claim correct? Explain your answer. (3 marks)

Solution.

(a) $\dfrac{QR}{\sin 95°} = \dfrac{30}{\sin(180° - 95° - 40°)}$ 1M

$QR = 42.26496158 = 42.3$ cm 1A

(b) $MQ = \dfrac{1}{2}QR = 21.13248079$

$PM^2 = PQ^2 + MQ^2 - 2(PQ)(MQ)\cos 40º$ 1M

$PM = 19.37205657 = 19.4$ cm 1A

(c) Let N a point on PQ such that $MN \perp PQ$.

$\sin 40º = \dfrac{MN}{MQ}$

$MN = 13.58369682$ 1A

$\sin 70º = \dfrac{MX}{MN}$

$MX = 12.76449966$ 1A

$\sin\angle MPX = \dfrac{MX}{MP}$

$\angle MPX = 41.2º$

Yes. The claim is agreed. 1A

7

19. The centre of the circle C is the point $G(83, 112)$. It is found that the point $A(158, 12)$ lies outside C. AP and AQ are the tangents to C at the points P and Q respectively. It is given that C passes through the point $(23, 67)$. [equations of circles]

(a) Find AG. (2 marks)

(b) Show that $APGQ$ is a cyclic quadrilateral. Hence, find the equation of the circumcircle of ΔAPQ. (4 marks)

(c) Find the equation of the straight line passing through P and Q. (4 marks)

(d) Someone claims that the circumcenter of ΔAPQ lies outside the circle C. Do you agree? Explain your answer. (2 marks)

Solution.

(a) $AG^2 = (158 - 83)^2 + (12 - 112)^2$ 1M

$AG = 125$ 1A

(b) $AP \perp GP$ and $AQ \perp GQ$

$\angle APG + \angle AQG = 180^\circ$

$APGQ$ is a cyclic quadrilateral. 1A

Note that AG is a diameter of the circumcircle of ΔAPQ. 1A

Equation of circumcircle of ΔAPQ is

$\frac{y-12}{x-158} \times \frac{y-112}{x-83} = -1$ 1A

$x^2 + y^2 - 241x - 124y + 14458 = 0$ 1A

(c) Let the intersection of AG and PQ be X.

$GP = \sqrt{(23-83)^2 + (67-112)^2} = 75$ 1A

$\frac{GX}{GP} = \frac{GP}{AG}$

$GX = \frac{75^2}{125} = 45$ 1A

$X = \left(\frac{(125-45)(83)+45(158)}{125}, \frac{(125-45)(112)+45(12)}{125}\right) = (110, 76)$ 1A

Equation of PQ is

$\frac{y-76}{x-110} \times \frac{112-12}{83-158} = -1$

$3x - 4y - 26 = 0$ 1A

(d) Distance of circumcenter of ΔAPQ from G is $\frac{AG}{2} = 62.5$ 1A

Radius of $C = GP = 75 > 62.5$

No, the claim is disagreed. 1A

(c) *Alternative Solution*

Radius of $C = \sqrt{(23-83)^2+(67-112)^2} = 75$ 1A

Equation of C is

$(x-83)^2+(y-112)^2=75^2$ 1M

$x^2+y^2-166x-224y+13808=0$ 1A

Equation of PQ is

$(x^2+y^2-166x-224y+13808)-(x^2+y^2-241x-124y+14458)=0$

$3x-4y-26=0$ 1A

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END OF PAPER