FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

SECTION A (50 marks)

1. Let $f(x) = \frac{3x}{x^2-1}$. Prove that $f(2+h) - f(2) = -\frac{5h+2h^2}{3+4h+h^2}$. Hence find $f'(2)$ from first principles. (4 marks)

$f(2+h) - f(2)$

$= \frac{3(2+h)}{(2+h)^2 - 1} - \frac{3(2)}{2^2-1}$ 1M

$= \frac{6+3h}{h^2+4h+3} - 2$

$= \frac{6+3h-6-8h-2h^2}{3+4h+h^2}$

$= \frac{-5h-2h^2}{3+4h+h^2}$

$= -\frac{5h+2h^2}{3+4h+h^2}$ 1f.t.

$f'(2)$

$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$ 1M

$= \lim_{h\to 0} -\frac{5h+2h^2}{3+4h+h^2} \times \frac{1}{h}$ (from above)

$= \lim_{h\to 0} -\frac{5+2h}{3+4h+h^2}$ * key step

$= -\frac{5+2(0)}{3+4(0)+(0)^2}$

$= -\frac{5}{3}$ 1A

2. Let $C: 9x^2+16y^2=25$ be a curve and $P(1, 1)$ is a point on C. Find the equation of normal to the curve C at P.

(4 marks)

$\frac{d}{dx}(9x^2+16y^2)=\frac{d}{dx}(25)$

$18x+32y\frac{dy}{dx}=0$ 1M

$\frac{dy}{dx}=-\frac{9x}{16y}$

$\therefore$ Slope of tangent at P

$=\frac{dy}{dx}\Big|_{P(1,1)}$ 1M

$=-\frac{9(1)}{16(1)}$

$=-\frac{9}{16}$

$\therefore$ Equation of normal at P:

$\frac{y-1}{x-1}\cdot\left(-\frac{9}{16}\right)=-1$ 1M

i.e. $16x-9y-7=0$ 1A

3. Let $P(x)=2(x-k)^7-12(x-k)^2+71(x-k)-36$ be a polynomial in x where $k\in\Re$.

(a) Prove that $P''(k)$ is independent on k.

(b) If the coefficient of x^4 is 560, find the value of k.

(4 marks)

(a) $P(x)=2(x-k)^7-12(x-k)^2+71(x-k)-36$

$P'(x)=14(x-k)^6-24(x-k)+71$ 1M

$P''(x)=84(x-k)^5-24$

$P''(k)=84(k-k)^5-24$

$=-24$ which is a constant independent on k. 1f.t.

(b) Coefficient of x^4

$=2\,C^7_4(-k)^{7-4}=560$ 1M

$\therefore (-k)^3=8$

$-k=2$

$k=-2$ 1A

4. (a) Using mathematical induction, prove that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for all positive integers n.

(b) Using (a), prove that $\sum_{k=n}^{2n} k^3 = \frac{3n^2(n+1)(5n+1)}{4}$ for all positive integers n.

(6 marks)

(a) When $n=1$,

$LHS = \sum_{k=1}^{1} k^3$

$= 1^3 = 1$

$RHS = \frac{(1)^2(1+1)^2}{4}$

$= 1 = LHS$

$\therefore$ It is true for $n=1$ 1 f.t.

Assume it is true for $n=r$ for some positive integer r.

i.e. $\sum_{k=1}^{r} k^3 = \frac{r^2(r+1)^2}{4}$ 1M

When $n=r+1$,

$LHS = \sum_{k=1}^{r+1} k^3$

$= \sum_{k=1}^{r} k^3 + (r+1)^3$

$= \frac{r^2(r+1)^2}{4} + (r+1)^3$ (I.A.) 1M

$= \frac{(r+1)^2}{4}\left[r^2 + 4(r+1)\right]$

$= \frac{(r+1)^2}{4}(r^2+4r+4)$

$\therefore LHS = \frac{(r+1)^2(r+2)^2}{4}$

$= \frac{(r+1)^2\left[(r+1)+1\right]^2}{4}$

$= RHS.$

$\therefore$ It is true for $n=r+1$ when it is true for $n=r$.

$\therefore$ By the principle of induction, $\sum_{k=1}^{n} k = \frac{n^2(n+1)^2}{4}$ is true for all positive integers n 1f.t.

(b) $\sum_{k=n}^{2n} k^3$

$= \sum_{k=1}^{2n} k^3 - \sum_{k=1}^{n-1} k^3$

$= \frac{(2n)^2(2n+1)^2}{4} - \frac{(n-1)^2(n-1+1)^2}{4}$ 1M

$= \frac{n^2}{4}\left[4(2n+1)^2 - (n-1)^2\right]$

$= \frac{n^2}{4}(16n^2+16n+4 - n^2+2n-1)$

$= \frac{n^2}{4}(3)(5n^2+6n+1)$

$= \frac{3n^2}{4}(n+1)(5n+1)$ 1A

5. A scientist performs an experiment to study the rate of growth of a plant X. The experiment lasts for 10 days. At the start of the experiment, the height of X is 2 cm. The scientist finds that during the experiment, $\frac{dH}{dt} = 6 - 2\sqrt{t}$, where H cm is the height of X and t is the number of days elapsed since the start of the experiment.

The scientist claims that X will reach its maximum height of 24 cm during the experiment. Is the claim correct? Explain your answer. (6 marks)

$\frac{dH}{dt} = 6 - 2\sqrt{t}$

$\therefore H = \int (6 - 2t^{\frac{1}{2}})\,dt$ 1M

$= 6t - 2(\frac{2}{3})t^{\frac{3}{2}} + c$

$= 6t - \frac{4}{3}t^{\frac{3}{2}} + c$

When $t = 0$, $H = 2$

$\therefore 2 = 6(0) - \frac{4}{3}(0)^{\frac{3}{2}} + c$ 1M

$c = 2$

$\therefore H = 6t - \frac{4}{3}t^{\frac{3}{2}} + 2$

When $\frac{dH}{dt} = 0$

$6 - 2\sqrt{t} = 0$

$\sqrt{t} = 3$

$t = 9$ (accepted as $0 \le t \le 10$) 1M

Note $\frac{d^2H}{dt^2} = -2(\frac{1}{2})t^{-\frac{1}{2}}$

$= -t^{-\frac{1}{2}}$

$\frac{d^2H}{dt^2}\Big|_{t=9} = -(9)^{-\frac{1}{2}} = -\frac{1}{3} < 0$ 1M

$\therefore$ max H attains at $t = 9$

and max H

$= 6(9) - \frac{4}{3}(9)^{\frac{3}{2}} + 2$ 1M

$= 20 \ne 24$

$\therefore$ No. 1f.t.

6. Let A be a real number such that $A \neq \left(n+\frac{1}{2}\right)\pi$ for any integer n.

(a) Prove that $\tan 3A = \frac{3t-t^3}{1-3t^2}$ where $t = \tan A$.

(b) Using the fact that $\tan\frac{3\pi}{5} + \tan\frac{2\pi}{5} = 0$, find the value of $\tan^2\frac{\pi}{5}$.

(6 marks)

(a) $\tan 3A$

$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$ 1M

$= \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A \cdot \frac{2\tan A}{1-\tan^2 A}}$

$= \frac{t + \frac{2t}{1-t^2}}{1 - \frac{2t^2}{1-t^2}} \cdot \frac{(1-t^2)}{(1-t^2)}$

$= \frac{(t-t^3)+2t}{1-t^2-2t^2}$

$= \frac{3t-t^3}{1-3t^2}$ 1f.t.

(b) $\tan\frac{3\pi}{5} + \tan\frac{2\pi}{5} = 0$

$\frac{3t-t^3}{1-3t^2} + \frac{2t}{1-t^2} = 0$ where $t = \tan\frac{\pi}{5}$ 1M

$\frac{t}{(1-3t^2)(1-t^2)}\left[(3-t^2)(1-t^2) + 2(1-3t^2)\right] = 0$

$\therefore t = 0$ (rej) or $t^4 - 4t^2 + 3 + 2 - 6t^2 = 0$ 1M

$t^4 - 10t^2 + 5 = 0$

$\therefore t^2 = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(5)}}{2(1)}$ 1M

$= 5 \pm 2\sqrt{5}$

Note $\frac{\pi}{5} < \frac{\pi}{4}$

$\therefore t = \frac{\pi}{5} < \tan\frac{\pi}{4} = 1$

$t^2 < 1$

$\therefore t^2 = 5 - 2\sqrt{5}$ or $5 + 2\sqrt{5} > 1$ (rej)

$\therefore \tan^2\frac{\pi}{5} = 5 - 2\sqrt{5}$ 1A

7. Let **a** and **b** be two vectors.

(a) Prove that $|\mathbf{a}\times\mathbf{b}|^2+(\mathbf{a}\cdot\mathbf{b})^2=|\mathbf{a}|^2|\mathbf{b}|^2$.

(b) Given that $|\mathbf{a}|=3$, $|\mathbf{b}|=5$ and $\mathbf{a}\cdot\mathbf{b}=12$.

(i) Find $|\mathbf{a}\times\mathbf{b}|$.

(ii) Define $\mathbf{P}_1=\mathbf{a}$ and $\mathbf{P}_{n+1}=\mathbf{P}_n+\mathbf{b}$ for any natural number n.

Find $|\mathbf{P}_{20219}\times\mathbf{P}_{2020}|$

(6 marks)

(a) LHS

$=|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2$

$=(|\vec{a}||\vec{b}|\sin\theta)^2+(|\vec{a}||\vec{b}|\cos\theta)^2$ where θ is the angle from $\vec{a}$ to $\vec{b}$ 1M

$=|\vec{a}|^2|\vec{b}|^2(\sin^2\theta+\cos^2\theta)$

$=|\vec{a}|^2|\vec{b}|^2$ 1f.t.

(b) (i) From (a),

$|\vec{a}\times\vec{b}|^2+(12)^2=(3)^2(5)^2$

$|\vec{a}\times\vec{b}|^2=81$

$\therefore |\vec{a}\times\vec{b}|=9$ (Note $|\vec{a}\times\vec{b}|\geq 0$) 1A

(ii) Similar to A.S.,

$\vec{P_n}=(n-1)\vec{b}+\vec{a}$ 1A

$\therefore |\overrightarrow{P_{20219}}\times\overrightarrow{P_{2020}}|$

$=|(20218\vec{b}+\vec{a})\times(2019\vec{b}+\vec{a})|$

$=|20218\cdot 2019\ \vec{b}\times\vec{b}+2019\vec{a}\times\vec{b}$ 1M

$+20218\vec{b}\times\vec{a}+\vec{a}\times\vec{a}|$

$=|2019\vec{a}\times\vec{b}-20218\vec{a}\times\vec{b}|$

$=|-18199\vec{a}\times\vec{b}|$

$=18199\times 9$ 1A

$=163791$

8. Let a, b, c and d be real numbers. Define $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $M^2 = M$. It is known that b and c are non-zero.

(a) Show that $a + d = 1$.

(b) Does M^{-1} exist ? Explain your answer.

(c) Prove that $(I - M)^n = I - M$ for any positive integers n where I is the 2×2 identity matrix.

(7 marks)

(a) $M^2 = M$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore \begin{cases} a^2+bc = a & \text{—— (1)} \\ ab+bd = b & \text{—— (2)} \\ ac+cd = c & \text{—— (3)} \\ bc+d^2 = d & \text{—— (4)} \end{cases}$ 1M

From (2) and note $b \neq 0$

$b(a+d) = b$

$\therefore a+d = 1$ 1f.t.

(b) Assume M^{-1} exists.

$M^{-1}(M^2) = M^{-1}(M)$

$M^{-1}(M \cdot M) = I$

$M = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 1M

which means $b=0$ and $c=0$ (contradiction arises)

$\therefore M^{-1}$ does not exist. 1f.t.

Alternative solution

(1) + (4), $a^2+bc+bc+d^2 = a+d$

$(a+d)^2 - 2ad + 2bc = a+d$ 1M

$1^2 - 2(ad-bc) = 1$ (from (a))

$ad - bc = 0$

$\therefore \det(M) = 0$

$\therefore M^{-1}$ does not exist. 1f.t.

(c) When $n=1$,

$LHS = (I-M)^1 = I-M = RHS$

$\therefore$ It is true for $n=1$ 1f.t.

Assume $(I-M)^k = I-M$ for some positive integer k.

When $n=k+1$,

$LHS = (I-M)^{k+1}$

$= (I-M)^k(I-M)$

$= (I-M)(I-M)$ (I.A.) 1M

$= I - M - M + M^2$

$= I - 2M + M$ ($\because M^2 = M$)

$= I - M = RHS$

$\therefore$ It is true for $n=k+1$ when it is true for $n=k$.

$\therefore$ By the principle of induction, $(I-M)^n = I-M$ for all positive integers n. 1ft

9. (a) Using integration by parts, prove that

$$\int u^2(2^u)\,du = \frac{u^2(2^u)}{\ln 2} - \frac{2u(2^u)}{(\ln 2)^2} + \frac{2^{u+1}}{(\ln 2)^3} + C,$$ where C is an integration constant.

(b) Let $f(x) = x(2^x)$ for all real numbers x and A be the area bounded by the graph of $y = f(x)$, the straight line $x = 1$ and the x-axis. Find the volume of the solid of revolution generated by revolving A about the x-axis.

(7 marks)

(a) $\int u^2(2^u)\,du$

$= \int u^2 e^{u\ln 2}\,du$

$= \int u^2\, d\left(\frac{e^{u\ln 2}}{\ln 2}\right)$

$= \frac{u^2 e^{u\ln 2}}{\ln 2} - \int \frac{e^{u\ln 2}}{\ln 2}\, d(u^2)$ 1M+1A

$= \frac{u^2(2^u)}{\ln 2} - \int \frac{2u e^{u\ln 2}}{\ln 2}\, du$

$= \frac{u^2(2^u)}{\ln 2} - \frac{2}{\ln 2}\int u\, d\left(\frac{e^{u\ln 2}}{\ln 2}\right)$

$= \frac{u^2(2^u)}{\ln 2} - \frac{2}{\ln 2}\left(\frac{u e^{u\ln 2}}{\ln 2} - \int \frac{e^{u\ln 2}}{\ln 2}\, du\right)$ 1M

$= \frac{u^2(2^u)}{\ln 2} - \frac{2u(2^u)}{(\ln 2)^2} + \frac{2}{(\ln 2)^2}\int e^{u\ln 2}\, du$

$= \frac{u^2(2^u)}{\ln 2} - \frac{2u(2^u)}{(\ln 2)^2} + \frac{2}{(\ln 2)^2}\cdot\frac{e^{u\ln 2}}{\ln 2} + C$

$= \frac{u^2(2^u)}{\ln 2} - \frac{2u(2^u)}{(\ln 2)^2} + \frac{2(2^u)}{(\ln 2)^3} + C$

$= \frac{u^2(2^u)}{\ln 2} - \frac{2u(2^u)}{(\ln 2)^2} + \frac{2^{u+1}}{(\ln 2)^3} + C$ 1ft.

(b) When $y=0$, $x(2^x)=0$

$x=0$ or $2^x=0$ (rej.)

$\therefore$ Required volume

$= \int_0^1 \pi[x(2^x)]^2\,dx$ ~~1M~~ * find lower limit

$= \pi\int_0^1 x^2 2^{2x}\,dx$

$= \pi\int_0^2 \left(\frac{u}{2}\right)^2 2^u \left(\frac{1}{2}du\right)$ where $u=2x$ 1M

$= \frac{\pi}{8}\int_0^2 u^2(2^u)\,du$

$= \frac{\pi}{8}\left[\frac{u^2(2^u)}{\ln 2} - \frac{2u(2^u)}{(\ln 2)^2} + \frac{2^{u+1}}{(\ln 2)^3}\right]_0^2$

$= \frac{\pi}{8}\left[\frac{2^2(2^2)}{\ln 2} - \frac{2(2)(2^2)}{(\ln 2)^2} + \frac{2^{2+1}-2^{0+1}}{(\ln 2)^3}\right]$

$= \frac{\pi}{4}\left[\frac{8}{\ln 2} - \frac{8}{(\ln 2)^2} + \frac{3}{(\ln 2)^3}\right]$ 1A

SECTION B (50 marks)

10. (a) Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x + y + z = a \\ -kx - y + kz = a^2 \\ k^2x + y - kz = a \end{cases}, \text{ where } a \in \Re .$$

(i) Assume that (E) has a unique solution.

(1) Find the range of values of k.

(2) Express the value of x in terms of a and k.

(ii) Assume that $k = -1$ and (E) is consistent.

(1) Find the value(s) of a.

(2) Solve (E).

(iii) Assume that $k = 1$ and (E) is not solvable. Find the range of values of a.

(8 marks)

(b) Consider the system of linear equations in real variables x, y, z

$$(F):\begin{cases} x + y + z = 3b \\ 2x - y - 2z = 9b^2 \\ 4x + y + 2z = 3b \\ 8x - 3y - 3z = c \end{cases}, \text{ where } b, c \in \Re .$$

Assume that (F) is consistent. Find the range of values of c.

(4 marks)

(a)(i)(1) (E) has a unique solution

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ -k & -1 & k \\ k^2 & 1 & -k \end{vmatrix} \neq 0 \qquad 1M$$

$$\begin{vmatrix} 0 & 1 & 0 \\ -k+1 & -1 & k+1 \\ k^2-1 & 1 & -k-1 \end{vmatrix} \neq 0 \qquad \begin{pmatrix} C_1 - C_2 \to C_1 \\ C_3 - C_2 \to C_3 \end{pmatrix}$$

$$(k+1)(k-1)\begin{vmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ k+1 & 1 & -1 \end{vmatrix} \neq 0$$

$$k(k+1)(k-1) \neq 0$$

$\therefore$ $k \neq 0$ and $k \neq -1$ and $k \neq 1$ 1A

(a)(i)(2) By Cramer's rule,

$$x = \frac{\begin{vmatrix} a & 1 & 1 \\ a^2 & -1 & k \\ a & 1 & -k \end{vmatrix}}{k(k+1)(k-1)} \qquad 1M$$

$$= \frac{\begin{vmatrix} a & 1 & 1 \\ a^2+a & 0 & k+1 \\ 0 & 0 & -(k+1) \end{vmatrix}}{k(k+1)(k-1)} \qquad \begin{pmatrix} R_2 + R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{pmatrix}$$

$$= \frac{(k+1)(a^2+a)}{k(k+1)(k-1)}$$

$$= \frac{a(a+1)}{k(k-1)}. \qquad 1A$$

(a)(ii)(1) Put $k=-1$,

$$(E) \sim \begin{pmatrix} 1 & 1 & 1 & a \\ 1 & -1 & -1 & a^2 \\ 1 & 1 & 1 & a \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & -2 & -2 & a^2-a \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{pmatrix}$$ share with (a)(i) 1M*

As the last equation is trivial for any values of a, a can take any real values. 1A

(2) Let $z = t \in \mathbb{R}$, $y = \frac{a-a^2}{2} - t$

$x = \frac{a^2+a}{2}$ 1A

(a)(iii) Put $k=1$,

$$(E) \sim \begin{pmatrix} 1 & 1 & 1 & a \\ -1 & -1 & 1 & a^2 \\ 1 & 1 & -1 & a \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 0 & 2 & a^2+a \\ 0 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} R_2+R_1 \to R_2 \\ R_3-R_1 \to R_3 \end{pmatrix}$$

Share with (a)(i)(1)

$$\sim \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 0 & 2 & a^2+a \\ 0 & 0 & 0 & a^2+a \end{pmatrix} (R_3+R_2 \to R_3) \quad 1M^*$$

As (E) is not solvable,

$a^2+a \neq 0$

$a(a+1) \neq 0$

$a \neq 0$ and $a \neq -1$ 1A

(b) Put $k=-2$, $a=3b$ into (E). 1M

(E) will be the same as the first 3 equations of (F).

by (a)(i)(2), $x = \frac{3b(3b+1)}{(-2)(-2-1)}$

$= \frac{3b^2+b}{2}$

Note in (F), (1st equation)$\times 3$ + (4th equation),

$11x = c+9b$ 1M

$\therefore \frac{11(3b^2+b)}{2} - 9b = c$

$c = \frac{33}{2}b^2 - \frac{7b}{2}$

$= \frac{33}{2}\left(b^2 - \frac{7}{33}b\right)$

$= \frac{33}{2}\left[\left(b-\frac{7}{66}\right)^2 - \left(\frac{7}{66}\right)^2\right]$ 1M

$c = \frac{33}{2}\left(b-\frac{7}{66}\right)^2 - \frac{49}{264} \geq -\frac{49}{264}$

$\therefore$ Required range : $c \geq -\frac{49}{264}$ 1A

11. Let $f(x) = \frac{3x}{x^3 - 1}$ where $x \neq 1$ and C be the graph of $y = f(x)$.

(a) Find all the asymptote(s) in C.

(2 marks)

(b) (i) Find $f'(x)$.

(ii) For $x > 1$, determine if $f(x)$ is decreasing.

(iii) Find the extreme point(s) of C.

(6 marks)

(c) (i) Show that $f(x)$ can be rewritten as $\frac{1}{x-1} - \frac{2x+1}{2(x^2+x+1)} + \frac{3}{2(x^2+x+1)}$.

(ii) Find the area enclosed by C, the x-axis and $x = -1$.

(5 marks)

(a) $\lim_{x\to\infty} \frac{3x}{x^3-1}$

$= \lim_{x\to\infty} \frac{\frac{3}{x^2}}{1-\frac{1}{x^3}}$

$= \frac{0}{1-0^3} = 0$

$\therefore$ horizontal asymptote : $y = 0$ 1A

Note $x^3 - 1 = 0$

$x^3 = 1$

$x = 1$

$\therefore$ vertical asymptote : $x = 1$. 1A

(b)(i) $f(x) = \frac{3x}{x^3-1}$

$f'(x) = \frac{(x^3-1)(3) - 3x(3x^2)}{(x^3-1)^2}$ 1M

$= \frac{-6x^3-3}{(x^3-1)^2}$ 1A

(ii) Note $(x^3-1)^2 > 0$ for any $x \neq 1$

and $-6x^3 - 3 < -9 < 0$ for any $x > 1$

$\therefore$ $f'(x) < 0$ for any $x > 1$

$\therefore$ $f(x)$ is decreasing 1ft.

(iii) $f'(x) = 0$

$-6x^3 - 3 = 0$

$x^3 = -\frac{1}{2}$

$x = -\frac{1}{\sqrt[3]{2}}$ 1M

x	$(-\infty, -\frac{1}{\sqrt[3]{2}})$	$(-\frac{1}{\sqrt[3]{2}}, 1)$	$(1, \infty)$
$f'(x)$	+ve	−ve	−ve

1M

$\therefore$ Relative maximum point of C attains at $x = -\frac{1}{\sqrt[3]{2}}$

Note $f(-\frac{1}{\sqrt[3]{2}})$

$= \frac{-\frac{3}{\sqrt[3]{2}}}{-\frac{1}{2} - 1}$

$= \sqrt[3]{4}$

$\therefore$ max pt of $C = (-2^{-\frac{1}{3}}, 2^{\frac{2}{3}})$ 1A

(c) (i) $\frac{1}{x-1}-\frac{2x+1}{2(x^2+x+1)}+\frac{3}{2(x^2+x+1)}$

$=\frac{2(x^2+x+1)-(2x+1)(x-1)+3(x-1)}{2(x-1)(x^2+x+1)}$

$=\frac{2(x^2+x+1)-(2x-2)(x-1)}{2(x-1)(x^2+x+1)}$

$=\frac{x^2+x+1-(x-1)^2}{(x-1)(x^2+x+1)}$

$=\frac{x^2+x+1-x^2+2x-1}{x^3-1}$

$=\frac{3x}{x^3-1}=f(x)$ 1 f.t.

(ii) When $f(x)=0$, $\frac{3x}{x^3-1}=0$

$x=0$

$\therefore$ Required area

$=\left|\int_{-1}^{0} f(x)\,dx\right|$ 1M upper limit ~~+ absolute sign~~

$=\left|\int_{-1}^{0}\left[\frac{1}{x-1}-\frac{2x+1}{2(x^2+x+1)}+\frac{3}{2(x^2+x+1)}\right]dx\right|$ (by (c)(i))

Note

$\int_{-1}^{0}\frac{1}{x-1}\,dx$

$=[\ln|x-1|]_{-1}^{0}$

$=-\ln 2$ 1M* (ln| |)

$\int_{-1}^{0}\frac{2x+1}{2(x^2+x+1)}\,dx$

$=\frac{1}{2}\int_{-1}^{0}\frac{1}{x^2+x+1}\,d(x^2+x+1)$

$=\frac{1}{2}\left[\ln|x^2+x+1|\right]_{-1}^{0}$ 1M* (ln| |)

$=0$

$\int_{-1}^{0}\frac{3}{2(x^2+x+1)}\,dx$

$=\frac{3}{2}\int_{-1}^{0}\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$

$=\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\left[\tan^{-1}\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]\right]_{-1}^{0}$ 1M ($\tan^{-1}$( ))

$=\sqrt{3}\left[\frac{\pi}{6}-(-\frac{\pi}{6})\right]$

$=\frac{\pi\sqrt{3}}{3}$

$\therefore$ Required area

$=\left|-\ln 2-0+\frac{\pi\sqrt{3}}{3}\right|$

$=\frac{\pi\sqrt{3}}{3}-\ln 2$ 1A

12. (a) Define $I_{m,n} = \int_{-1}^{1} (1+x)^m (1-x)^n \, dx$ where m, n are real numbers.

(i) Express $I_{m,0}$ in terms of m where $m \neq -1$.

(ii) By considering $\frac{d}{dx}(1+x)^{m+1}(1-x)^n$, show that $I_{m,n} = \frac{n}{m+1} I_{m+1,n-1}$ for any $n \neq 0$ and $m \neq -1$.

(iii) Hence evaluate $\int_{-1}^{1} (1+x)^{10}(1-x)^3 \, dx$.

(6 marks)

(b) Define $J_{p,q} = \int_0^{\frac{\pi}{2}} \cos^p \theta \sin^q \theta \, d\theta$ where p, q are real numbers.

(i) Using integration by substitution, show that $I_{m,n} = 2^{m+n+2} J_{2m+1,\,2n+1}$.

(ii) Using (a)(ii) and (b)(i), deduce that $J_{p,q} = \frac{q-1}{p+1} J_{p+2,\,q-2}$ where $p \neq -1$.

(iii) Using (b)(ii), evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta} \sin^5 \theta \, d\theta$

(7 marks)

(a)(i) $I_{m,0}$

$= \int_{-1}^{1} (1+x)^m (1-x)^0 \, dx$

$= \int_{-1}^{1} (1+x)^m \, dx$

$= \left[\frac{(1+x)^{m+1}}{m+1}\right]_{-1}^{1}$

$= \frac{2^{m+1}}{m+1}$ 1A

(ii) $\frac{d}{dx}(1+x)^{m+1}(1-x)^n$

$= (m+1)(1+x)^m(1-x)^n + n(1-x)^{n-1}(-1)(1+x)^{m+1}$

$= (m+1)(1+x)^m(1-x)^n - n(1+x)^{m+1}(1-x)^{n-1}$ 1A

$\therefore \int_{-1}^{1} (m+1)(1+x)^m(1-x)^n dx - \int_{-1}^{1} n(1+x)^{m+1}(1-x)^{n-1} dx$

$= \left[(1+x)^m(1-x)^n\right]_{-1}^{1}$ 1M

$(m+1) I_{m,n} - n I_{m+1,n-1} = 0$

$\therefore I_{m,n} = \frac{n}{m+1} I_{m+1,n-1}$ 1f.t.

(iii) $\int_{-1}^{1} (1+x)^{10}(1-x)^3 dx$

$= I_{10,3}$ Share with (b)(iii)

$= \frac{3}{11} I_{11,2}$ 1M*

$= \frac{3}{11} \cdot \frac{2}{12} I_{12,1}$

$= \frac{3}{11} \cdot \frac{2}{12} \cdot \frac{1}{13} I_{13,0}$

$= \frac{3}{11} \cdot \frac{2}{12} \cdot \frac{1}{13} \cdot \frac{2^{14}}{14}$

$= \frac{4096}{1001}$ 1A

(b)(i) Let $x = \cos 2\theta$ $\therefore dx = -2\sin 2\theta \, d\theta$ 1M

When $x=1$, $1 = \cos 2\theta \Rightarrow 2\theta = 0 \Rightarrow \theta = 0$

$x=-1$, $-1 = \cos 2\theta \Rightarrow 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$

$\therefore I_{m,n}$

$= \int_{\frac{\pi}{2}}^{0} (1+\cos 2\theta)^m (1-\cos 2\theta)^n (-2\sin 2\theta) d\theta$ useful

$= -2\int_{\frac{\pi}{2}}^{0} (2\cos^2\theta)^m (2\sin^2\theta)^n (2\sin\theta\cos\theta) d\theta$ 1M (trig formula)

$= 2\int_{0}^{\frac{\pi}{2}} 2^m \cos^{2m}\theta \, 2^n \sin^{2n}\theta \, 2\sin\theta\cos\theta \, d\theta$

$= 2^{m+n+2} \int_0^{\frac{\pi}{2}} \cos^{2m+1}\theta \sin^{2n+1}\theta \, d\theta$

$= 2^{m+n+2} J_{2m+1,2n+1}$ 1f.t.

(b)(ii) Put $m=\frac{p-1}{2}$, $n=\frac{q-1}{2}$ into (b)(i), 1M

$I_{m,n}=2^{\frac{p-1}{2}+\frac{q-1}{2}+2}J_{2(\frac{p-1}{2})+1,2(\frac{q-1}{2})+1}$

$=2^{\frac{p+q}{2}+1}J_{p,q}$

Similarly, $I_{m+1,n-1}=2^{(\frac{p-1}{2}+1)+(\frac{q-1}{2}-1)+2}J_{2(\frac{p-1}{2}+1)+1,\ 2(\frac{q-1}{2}-1)+1}$

$=2^{\frac{p+q}{2}+1}J_{p+2,q-2}$

Using (a)(ii),

$2^{\frac{p+q}{2}+1}J_{p,q}=\frac{\frac{q-1}{2}}{\frac{p-1}{2}+1}\cdot 2^{\frac{p+q}{2}+1}J_{p+2,q-2}$

$J_{p,q}=\frac{q-1}{p-1+2}J_{p+2,q-2}$

$=\frac{q-1}{p+1}J_{p+2,q-2}$ 1ft.

(iii) $\int_0^{\pi/2}\sqrt{\cos\theta}\sin^5\theta\,d\theta$

$=J_{\frac{1}{2},5}$

$=\frac{5-1}{\frac{1}{2}+1}J_{\frac{5}{2},3}$ 1M* share with (a)(ii)

$=\frac{8}{3}\cdot\frac{3-1}{\frac{5}{2}+1}J_{\frac{9}{2},1}$

$=\frac{32}{21}\int_0^{\pi/2}(\cos\theta)^{\frac{9}{2}}\sin\theta\,d\theta$

$=\frac{32}{21}\int_0^{\pi/2}(\cos\theta)^{9/2}(-d\cos\theta)$ 1M

$=\frac{-32}{21}\cdot\frac{2}{11}\left[(\cos\theta)^{11/2}\right]_0^{\pi/2}=\frac{64}{231}$. 1A

13. Let $\overrightarrow{OA} = 5\mathbf{i} + \mathbf{j} - 6\mathbf{k}$, $\overrightarrow{OB} = h\mathbf{i} + (1-h)\mathbf{j} + (3-2h)\mathbf{k}$, $\overrightarrow{OC} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OD} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, where O is the origin and h is a constant. It is given that $\overrightarrow{CD}$ is a vector perpendicular to a triangular plane OAB.

(a) (i) Find the value of h.

(ii) Find the area of ΔOAB

(4 marks)

(b) Let X be the foot of the perpendicular of C on the plane OAB.

(i) Find $\overrightarrow{OX}$.

(ii) Find the shortest distance between CD and OB.

(8 marks)

(a)(i) $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -2\vec{i} - 2\vec{j} - 2\vec{k}$

As $\overrightarrow{CD} \perp \overrightarrow{OB}$,

$(-2\vec{i} - 2\vec{j} - 2\vec{k}) \cdot [h\vec{i} + (1-h)\vec{j} + (3-2h)\vec{k}] = 0$ 1M

$2h + 2 - 2h + 6 - 4h = 0$

$h = 2$ 1A

(ii) $\therefore \overrightarrow{OB} = 2\vec{i} - \vec{j} - \vec{k}$

$\overrightarrow{OA} \times \overrightarrow{OB}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 1 & -6 \\ 2 & -1 & -1 \end{vmatrix}$

$= -7\vec{i} - 7\vec{j} - 7\vec{k}$

$\therefore$ Area of ΔOAB

$= \frac{1}{2} | \overrightarrow{OA} \times \overrightarrow{OB} |$ 1M

$= \frac{1}{2}(7)\sqrt{1^2+1^2+1^2}$

$= \frac{7\sqrt{3}}{2}$ 1A

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remarks (1) vector projection   orthogonal vector remarks(2)

(b)(i) Note $\overrightarrow{OX}$

$= \overrightarrow{OC} - \frac{\overrightarrow{OC} \cdot \overrightarrow{OA} \times \overrightarrow{OB}}{|\overrightarrow{OA} \times \overrightarrow{OB}|^2} \cdot (\overrightarrow{OA} \times \overrightarrow{OB})$ 1M + 1M

$= 3\vec{i} + \vec{j} + 3\vec{k} - \frac{3(-7) + (-7) + 3(-7)}{(7\sqrt{3})^2}(-7\vec{i} - 7\vec{j} - 7\vec{k})$ 1A

$= \frac{2}{3}\vec{i} - \frac{4}{3}\vec{j} + \frac{2}{3}\vec{k}$ 1A

(ii) Since $\overrightarrow{CX}$ and $\overrightarrow{CD}$ are both perpendicular to ΔOAB, C, X, D are collinear

$\therefore$ Required distance

= shortest distance between CX and OB

= shortest distance of X from $\overrightarrow{OB}$ 1M remarks(3)

$= \left| \overrightarrow{OX} - \frac{\overrightarrow{OX} \cdot \overrightarrow{OB}}{|\overrightarrow{OB}|^2} \overrightarrow{OB} \right|$ 1M

$= \left| \frac{2}{3}\vec{i} - \frac{4}{3}\vec{j} + \frac{2}{3}\vec{k} - \frac{\frac{4}{3} + \frac{4}{3} - \frac{2}{3}}{2^2 + (-1)^2 + (-1)^2}(2\vec{i} - \vec{j} - \vec{k}) \right|$ 1A

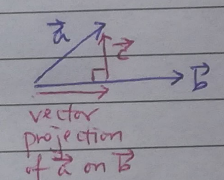
$= |-\vec{j} + \vec{k}|$

$= \sqrt{2}$ 1A

** Remarks

(1) Vector projection of $\vec{a}$ on $\vec{b}$

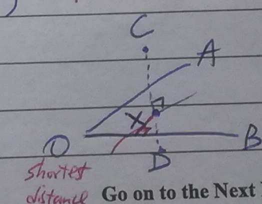
$= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$



(2) $\vec{c}$ (Note $\vec{c}$ is orthogonal to $\vec{b}$)

$= \vec{a}$ − vector projection

(3) shortest distance = X from OB



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