F.6 Math Mock Exam Paper 1 (Solutions)

1. $$\frac{(m^{\frac{-1}{2}}n^3)^2}{(n^{\frac{-2}{3}}m^4)^{-3}}$$
$$=\frac{m^{-1}n^6}{n^2m^{-12}}$$
$$=m^{11}n^4$$

2. $$k(2kx-h)=2x-h$$
$$2k^2x-hk=2x-h$$
$$2x(k^2-1)=h(k-1)$$
$$x=\frac{h(k-1)}{2(k+1)(k-1)}$$
$$x=\frac{h}{2(k+1)}$$

3. (a) $$3x^3-13x^2+12x$$
$$=x(3x^2-13x+12)$$
$$=x(3x-4)(x-3)$$

(b) $$(3x-4)^2+3x^3-13x^2+12x$$
$$=(3x-4)^2+x(3x-4)(x-3)$$
$$=(3x-4)(3x-4+x^2-3x)$$
$$=(3x-4)(x^2-4)$$
$$=(3x-4)(x+2)(x-2)$$

4. (a) $$\frac{2(2-x)}{-3}<4x+5$$
$$2x-4<12x+15$$
$$-19<10x$$
$$x>-\frac{19}{10}$$

(b) $5x - 17 \le 0$

$x \le \frac{17}{5}$

$\therefore -\frac{19}{10} < x \le \frac{17}{5}$

The integers satisfying the above inequalities are –1, 0, 1, 2 and 3.

5. (a) The maximum absolute error $=$ 5 mL

The least possible volume $=$ 870 mL – 5 mL $=$ 865 mL (0.865 L)

(b) The least possible amount of juice per each child $= \frac{865}{18} = 48.05...$

$= 50$ mL (cor. to the nearest 5 mL)

$\therefore$ It is possible.

6. Suppose x mL of wine A is mixed with y mL of wine B to form $(x + y)$ mL of wine C.

$0.1x + 0.4y = 0.2(x + y)$

$0.2y = 0.1x$

$\frac{x}{y} = \frac{2}{1}$

Therefore, the required ratio is $2:1$.

7. (a) $A' = (2, -3)$ $B' = (-4, 6)$

(b) slope of $OA' = \frac{-3-0}{2-0} = -\frac{3}{2}$

slope of $OB' = \frac{6-0}{-4-0} = -\frac{3}{2} =$ slope of OA'

$\therefore A', O, B'$ are collinear.

8. (a) $s = \sqrt{7^2 + 24^2} = 25$

(b) <u>Method 1</u>

area of sector = curved surface area of cone

$$\pi \cdot 25^2 \cdot \frac{\theta}{360°} = \pi \cdot 7 \cdot 25$$

$$\theta = 100.8°$$

Method 2

arc length of sector = base circumference of cone

$$2\pi \cdot 25 \cdot \frac{\theta}{360°} = 2\pi \cdot 7$$

$$\theta = 100.8°$$

9. (a) $85 - a = 42$

$a = 43$

$b - 30 = 67$

$b = 97$

(b) If the scores are arranged in ascending order, the 13th score is still 59, which is the median.

The median is unchanged. The claim is disagreed.

10. (a) Let $f(x) = ax^2 + bx$, where a and b are non–zero constants.

$f(3) = 9a + 3b = 15$ and $f(-5) = 25a - 5b = 55$

$3a + b = 5......(1)$ and $5a - b = 11......(2)$

$(1) + (2)$ $8a = 16$, $a = 2$

Put $a = 2$ in (1), $b = 5 - 3(2) = -1$

Thus, $f(x) = 2x^2 - x$.

(b)
$$\begin{aligned} y &= f(x) + 3 \\ &= 2x^2 - x + 3 \\ &= 2\left(x^2 - \frac{1}{2}x\right) + 3 \end{aligned}$$

$$= 2\left(x - \frac{1}{4}\right)^2 - \frac{1}{8} + 3$$

$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{23}{8}$$

Thus, the coordinates of the required vertex are $\left(\frac{1}{4}, \frac{23}{8}\right)$.

11. (a) Let $f(x) = (ax + b)(3x^2 + 4x - 5)$ where a and b are constants.

$f(2) = (2a + b)[3(2)^2 + 4(2) - 5]$

$30 = 15(2a + b)$

$2a + b = 2 ......(1)$

$$f\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}a + b\right)\left[3\left(-\frac{3}{2}\right)^2 + 4\left(-\frac{3}{2}\right) - 5\right]$$

$$51 = \left(-\frac{3}{2}a + b\right)\left(-\frac{17}{4}\right)$$

$$-\frac{3}{2}a + b = -12 ......(2)$$

$(1) - (2)$ $\qquad \frac{7}{2}a = 14 \qquad a = 4$

Put $a = 4$ in (1) $\qquad b = 2 - 2(4) = -6$

$\therefore$ the required quotient is $4x - 6$.

(b) $f(x) = 0$

$(4x - 6)(3x^2 + 4x - 5) = 0$

$x = \frac{3}{2}$ or $x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2(3)}$

$x = \frac{3}{2}$ or $x = \frac{-2 \pm \sqrt{19}}{3}$, which are irrational

$f(x) = 0$ has only one rational root $\frac{3}{2}$.

12. (a) 2

(b) mean

$= 1(10\%) + 2(45\%) + 3(1 - 10\% - 45\% - 15\%) + 4(15\%)$

$= 2.5$

(c) standard deviation

$$= \sqrt{10\%(1-2.5)^2 + 45\%(2-2.5)^2 + 30\%(3-2.5)^2 + 15\%(4-2.5)^2}$$

$$= \sqrt{0.75}$$

$$= \frac{\sqrt{3}}{2} = 0.866 \text{ (cor. to 3 sig. fig.)}$$

(d) <u>Method 1</u>

The required probability

$$= \frac{C_2^{90}}{C_2^{200}} = \frac{801}{3980}$$

<u>Method 2</u>

The required probability

$$= \frac{C_2^{60} + C_1^{60}C_1^{30} + C_2^{30}}{C_2^{200}} = \frac{801}{3980}$$

<u>Method 3</u>

The required probability

$$= \frac{90}{200} \times \frac{89}{199} = \frac{801}{3980}$$

<u>Method 4</u>

The required probability

$$= \frac{60}{200} \times \frac{59}{199} + \frac{60}{200} \times \frac{30}{199} + \frac{30}{200} \times \frac{60}{199} + \frac{30}{200} \times \frac{29}{199}$$

$$= \frac{801}{3980}$$

13. (a) Method 1

$\because \angle QOR = 90°$,

$\therefore QR$ is a diameter of the circle C. (converse of $\angle$ in semi–circle)

centre of circle $C = \left(\frac{6+0}{2}, \frac{-10+0}{2} \right) = (3, -5)$

radius $= \sqrt{(3-0)^2 + (-5-0)^2} = \sqrt{34}$

equation of circle C: $(x-3)^2 + (y+5)^2 = 34$ (or $x^2 + y^2 - 6x + 10y = 0$)

Method 2

Let the equation of the circle C be $x^2 + y^2 + Dx + Ey + F = 0$.

Put $(x, y) = (0, 0)$, $\therefore F = 0$.

Put $(x, y) = (6, 0)$, $\therefore 36 + 6D = 0$, $D = -6$

Put $(x, y) = (0, -10)$, $\therefore 100 - 10E = 0$, $E = 10$

$\therefore$ equation of the circle C is $x^2 + y^2 - 6x + 10y = 0$.

Method 3

Equation of the $\perp$ bisector of OQ is $x = \frac{6+0}{2} = 3$.

Equation of the $\perp$ bisector of OR is $y = \frac{-10+0}{2} = -5$.

The circumcentre of ΔOQR is the point of intersection of the $\perp$ bisectors, which is $(3, -5)$.

radius $= \sqrt{(3-0)^2 + (-5-0)^2} = \sqrt{34}$

equation of circle C: $(x-3)^2 + (y+5)^2 = 34$ (or $x^2 + y^2 - 6x + 10y = 0$)

(b) Let L be the tangent to the circle C at the point Q.

slope of $L \times$ slope of $QR = -1$ (tangent $\perp$ radius)

slope of $L \times \left(\dfrac{-10-0}{0-6} \right) = -1$

slope of $L = -\dfrac{3}{5}$

equation of L: $\dfrac{y-0}{x-6} = -\dfrac{3}{5}$

$5y = -3x + 18$

$3x + 5y - 18 = 0$ (or $y = -\dfrac{3}{5}x + \dfrac{18}{5}$)

(c) Let A be the centre of the circle C.

$\because$ tangent $\perp$ radius,

$\therefore PA^2 = 4^2 + (\sqrt{34})^2$ (Pyth. Thm.)

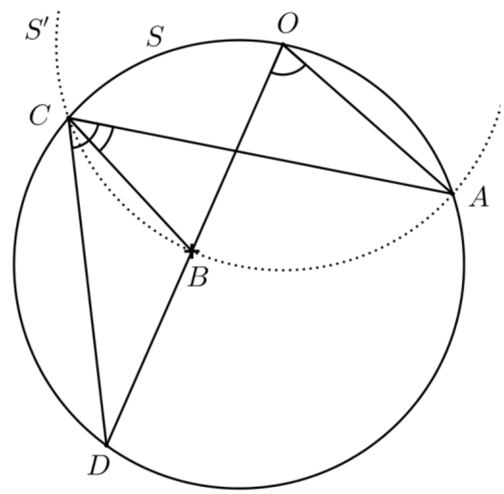
$PA^2 = 50$

$PA = 5\sqrt{2}$

$\because PA$ is a constant,

$\therefore$ locus of P is a circle with centre A, radius $5\sqrt{2}$.

14. (a) (i)



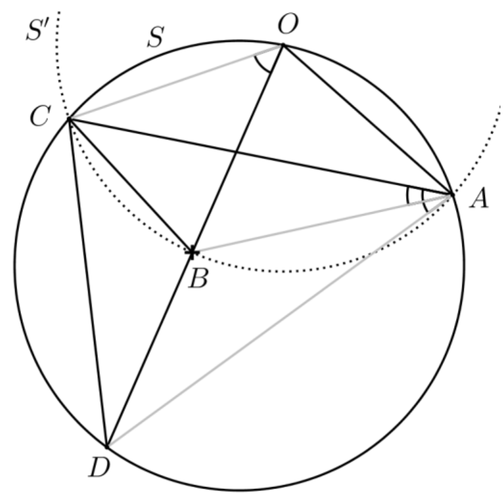
$\because \quad 2\angle BCA = \angle DOA$ (∠ at centre = 2 ∠ at ⊙ce in S')

$= \angle DCA$ (∠s in same segment; in S)

$= \angle DCB + \angle BCA$

$\therefore \quad \angle DCB = \angle BCA$

(ii)



Construct OC, AB and AD.

$\because \quad 2\angle CAB = \angle COD$ (∠ at centre = 2 ∠ at ⊙ce in S')

$= \angle CAD$ (∠s in same segment; in S)

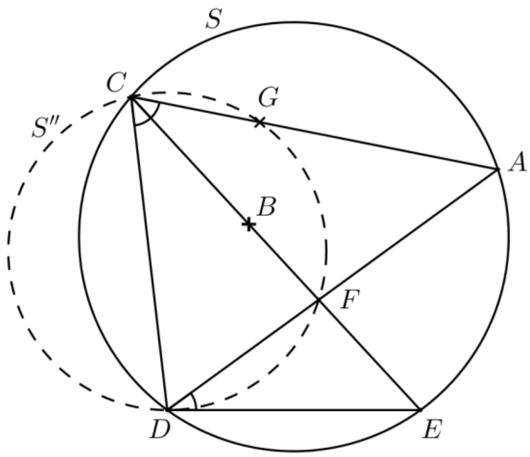
$= \angle CAB + \angle BAD$

$\therefore \quad \angle CAB = \angle BAD$

$\because$ $\angle CAB = \angle BAD$ (proved) and $\angle DCB = \angle BCA$ (by (a)(i))

$\therefore$ B is the in-centre of ΔACD.

(b) (i)



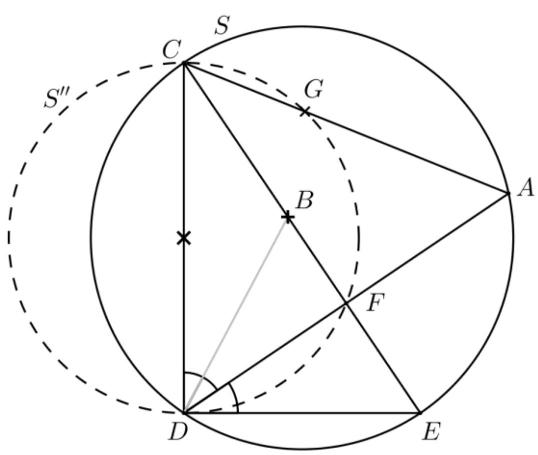
$\because$ $\angle DCE = \angle ECA$ (by (a)(i))

$= \angle EDA$ ($\angle$s in same segment; in S)

$\therefore$ DE is tangent to S'' at D.

(converse of $\angle$ in alt. segment; in S'')

(ii)



$\angle ADC = 2\angle CDB = 40°$ (by (a)(ii))

$\angle ADE = 90° - \angle ADC = 50°$ (tangent $\perp$ radius; in S'')

15. (a) Method 1 number of ways $= C_4^7 \times 2^4 = 560$

Method 2 number of ways $= \dfrac{14\times 12\times 10\times 8}{4!} = 560$

Method 3 number of ways $= C_4^7 \times 2 + C_3^7 \times C_1^4 + C_2^7 \times C_2^5 + C_1^7 \times C_3^6 = 560$

(b) Method 1

required probability $= \dfrac{C_2^7}{C_4^{14} - 560} = \dfrac{1}{21}$

Method 2

required probability $= \dfrac{C_2^7}{C_2^7 + C_3^7 \times 3 \times 2^2} = \dfrac{1}{21}$

16. Method 1

$$\begin{cases} 1 = b + \log_c 256 \\ 0 = b + \log_c 64 \end{cases}$$

$$\begin{cases} 1 - b = \log_c 256 \\ -b = \log_c 64 \end{cases}$$

$$\begin{cases} c^{1-b} = 256 \\ c^{-b} = 64 \end{cases}$$

$$\begin{cases} 64c = 256 \\ c^{-b} = 64 \end{cases}$$

$$\begin{cases} c = 4 \\ b = -3 \end{cases}$$

$y = -3 + \log_4 x$

$y + 3 = \log_4 x$

$x = 4^{y+3}$

Method 2

$y = b + \log_c x$

$y - b = \log_c x$

$x = c^{y-b}$

$$\begin{cases} 256 = c^{1-b} \\ 64 = c^{-b} \end{cases}$$

$$\begin{cases} 64c = 256 \\ c^{-b} = 64 \end{cases}$$

$$\begin{cases} c = 4 \\ b = -3 \end{cases}$$

$x = 4^{y+3}$

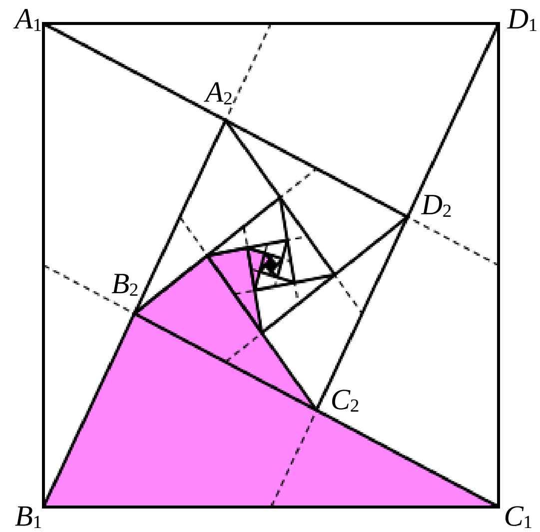
Method 3

Put $(x, y) = (64, 0)$	$b + \log_c 64 = 0$......(1)
Put $(x, y) = (256, 1)$	$b + \log_c 256 = 1$......(2)
$(2) - (1)$	$\log_c 256 - \log_c 64 = 1$
	$\log_c (256 \div 64) = 1$
	$\log_c 4 = 1$
	$c = 4$
Put $c = 4$ into (1)	$b = -\log_4 64 = -3$
	$\therefore y = -3 + \log_4 x$
	$y + 3 = \log_4 x$
	$x = 4^{y+3}$

17. (a) $B_1B_2 = 10\cos 60° = 5$ cm

$B_2C_1 = 10\sin 60° = 5\sqrt{3}$ cm

area of $\Delta C_1B_1B_2 = \frac{5(5\sqrt{3})}{2} = \frac{25\sqrt{3}}{2}$ cm^2



(b) $B_2C_2 = B_1C_1 \sin 60° - B_1C_1 \cos 60°$

$= \frac{B_1C_1(\sqrt{3}-1)}{2}$

$= 5(\sqrt{3}-1)$ cm

(c) In general, $\frac{B_nC_n}{B_{n-1}C_{n-1}} = \frac{\sqrt{3}-1}{2}$

$\frac{K_n}{K_{n-1}} = \left(\frac{B_nC_n}{B_{n-1}C_{n-1}}\right)^2 = \left(\frac{\sqrt{3}-1}{2}\right)^2$

$= \frac{3-2\sqrt{3}+1}{4} = \frac{4-2\sqrt{3}}{4} = 1-\frac{\sqrt{3}}{2}$

$\therefore K_1, K_2, \cdots, K_n, \cdots$ is a geometric sequence.

The area of the shaded regions

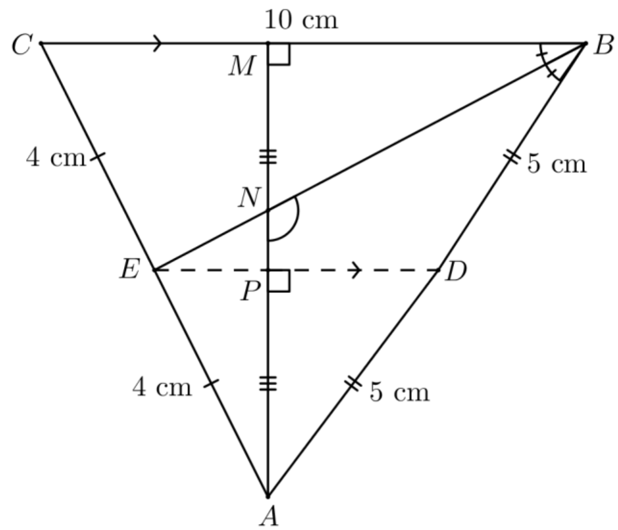
$= K_1 + K_2 + K_3 + \cdots$

$= \frac{25\sqrt{3}}{2} \times \frac{1}{1-\left(1-\frac{\sqrt{3}}{2}\right)}$

$= \frac{25\sqrt{3}}{2} \times \frac{1}{\left(\frac{\sqrt{3}}{2}\right)}$

$= 25$ cm^2

18.



(a) (i) $\cos\angle B = \dfrac{AB^2 + BC^2 - AC^2}{2AB\cdot BC} = \dfrac{10^2 + 10^2 - 8^2}{2\cdot 10\cdot 10} = \dfrac{17}{25}$

(ii) Length of required altitude

$= AM$, where M is the projection of A onto BC

$= AB\sin\angle B = AB\sqrt{1-\cos^2\angle B}$

$$= 10\sqrt{1-\left(\frac{17}{25}\right)^2} = \frac{8}{5}\sqrt{21} \text{ cm}$$

(b) (i) $CM = \sqrt{AC^2 - AM^2} = \sqrt{64 - \dfrac{64}{25}\cdot 21} = \dfrac{16}{5}$ cm

$\because$ $\Delta EPN \sim \Delta BNM$ (AAA)

$\therefore$ $\dfrac{EN}{NB} = \dfrac{EP}{MB}$ (corr. sides, ~ Δs)

$EN = EB\cdot\dfrac{EP}{EP+MB}$

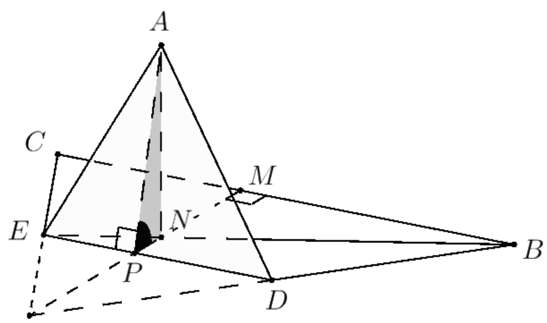
$= \sqrt{BC^2 - CE^2}\cdot\dfrac{\frac{1}{2}CM}{CB - \frac{1}{2}CM}$ (prop. of isos. Δ; intercept & mid-pt. thms.)

$$= \sqrt{100-16} \cdot \frac{\frac{8}{5}}{10-\frac{8}{5}} = \frac{8}{\sqrt{21}} \text{ cm}$$

$$AN = \sqrt{AE^2 - EN^2} = \sqrt{16 - \frac{64}{21}}$$

$$= 4\sqrt{\frac{17}{21}} \text{ cm} = 3.5986 \ldots \text{ cm} = 3.60 \text{ cm} \quad \text{(cor. to 3 sig. fig.)}$$

(ii)



Required angle $= \angle APN$,

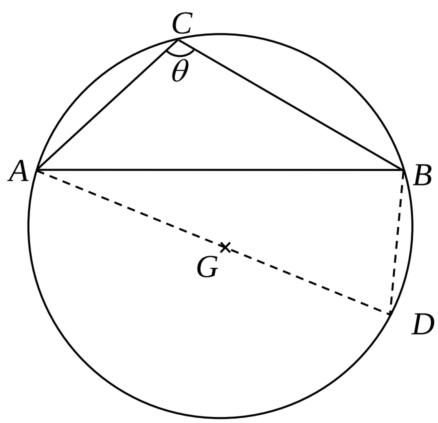
where P is the intersection of AN and DE <u>in Fig.4</u>.

$$\sin \angle APN = \frac{AN}{AP} = \frac{AN}{\frac{1}{2}AM} \quad \text{(mid–pt. \& intercept Thm.)}$$

$$= \frac{4\sqrt{\frac{17}{21}}}{\frac{4}{5}\sqrt{21}} = \frac{5\sqrt{17}}{21}$$

$$\angle APN = 79.019...° = 79.0° \quad \text{(cor. to 3 sig. fig.)}$$

19. (a)



(i) $\angle ADB + \angle ACB = 180^\circ$ (opp. $\angle$s, cyclic quad.)

$\angle ADB = 180^\circ - \theta$

$\angle ABD = 90^\circ$ ($\angle$ in semi–circle)

In ΔABD,

$$\frac{AB}{AD} = \sin \angle ADB$$

$$\frac{c}{2R} = \sin(180^\circ - \theta)$$

$$\sin\theta = \frac{c}{2R}$$

(ii) area of ΔABC

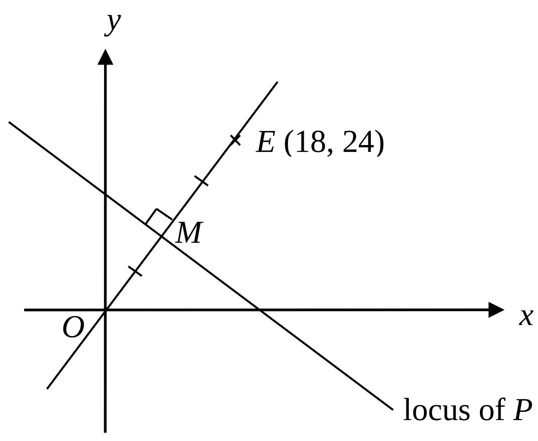
$$= \frac{1}{2}(AC)(BC)\sin \angle ACB$$

$$= \frac{1}{2}\cdot b\cdot a\cdot \sin\theta$$

$$= \frac{ab}{2}\cdot\frac{c}{2R}$$

$$= \frac{abc}{4R}$$

(b) (i)



Let M be the mid–point of OE.

$$M = \left(\frac{18+0}{2}, \frac{24+0}{2}\right) = (9, 12)$$

$$m_{OE} = \frac{24-0}{18-0} = \frac{4}{3}$$

$\therefore$ equation of $\perp$ bisector of OE:

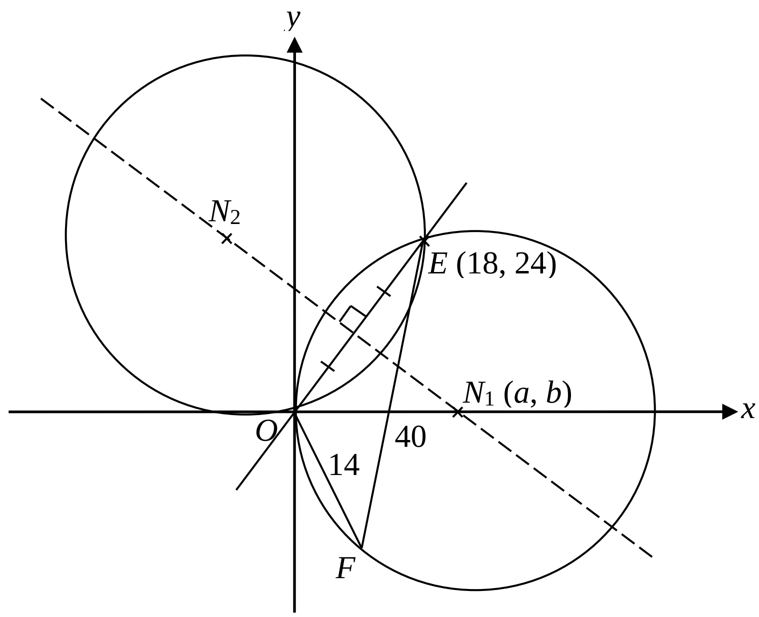
$$\frac{y-12}{x-9} = -1 \div \frac{4}{3}$$

$$\frac{y-12}{x-9} = -\frac{3}{4}$$

$$4y - 48 = -3x + 27$$

$$3x + 4y - 75 = 0 \qquad \left(\text{or } y = -\frac{3}{4}x + \frac{75}{4}\right)$$

(ii)



(I) $OE = \sqrt{(18-0)^2 + (24-0)^2} = 30$

semi–perimeter of $\Delta OEF = \dfrac{14+30+40}{2} = 42$

Let R_1 be the radius of the circumcircle of ΔOEF .

area of ΔOEF :

$\sqrt{42(42-14)(42-30)(42-40)} = \dfrac{14\times 30\times 40}{4R_1}$ (by (a)(ii))

$168 = \dfrac{16800}{4R_1}$

$R_1 = 25$

(II) $ON = R_1$

$(a-0)^2 + (b-0)^2 = 25^2$

$a^2 + b^2 = 625$......(1)

$\because$ the circumcentre of ΔOEF lies on the $\perp$ bisector of OE,

$\therefore$ by (b)(i), $3a + 4b - 75 = 0$

Method 1

$$a = -\frac{4}{3}b + 25 \text{......(1)}$$

$$\because OG = R_1,$$

$$\therefore a^2 + b^2 = 25^2 \text{......(2)}$$

Sub. (1) into (2)

$$\left(-\frac{4}{3}b + 25\right)^2 + b^2 = 625$$

$$\frac{16}{9}b^2 - \frac{200}{3}b + 625 + b^2 = 625$$

$$\frac{25}{9}b^2 - \frac{200}{3}b = 0$$

$$b^2 - 24b = 0$$

$$b(b - 24) = 0$$

$b = 0$ or 24

When $b = 0$, $a = 25$.

When $b = 24$, $a = -\frac{4}{3} \times 24 + 25 = -7$

Method 2

$$b = \frac{75 - 3a}{4} \text{......(1)}$$

$$\because OG = R_1,$$

$$\therefore a^2 + b^2 = 25^2 \text{......(2)}$$

Sub. (1) into (2)

$$a^2 + \left(\frac{75 - 3a}{4}\right)^2 = 625$$

$$a^2 + \frac{5625 - 450a + 9a^2}{16} = 625$$

$$16a^2 + 5625 - 450a + 9a^2 = 10000$$

$$25a^2 - 450a - 4375 = 0$$

$$a^2 - 18a - 175 = 0$$

$$(a - 25)(a + 7) = 0$$

$a = 25$ or -7

When $a = 25$, $b = \frac{75 - 3\times 25}{4} = 0$.

When $a = -7$, $b = \frac{75 - 3(-7)}{4} = 24$

(III) Method 1

area of ΔOEQ:

$$\frac{1}{2}\times QO \times QE \times \sin\angle OQE = 168$$

For minimum value of $QO \times QE$, $\sin\angle OQE$ should be maximum, which is 1.

Note that $\angle OQE$ can be 90° as the distance between the locus of Q and the line segment $OE \le OF = 14 < 15 = \frac{1}{2}OE$.

$\therefore$ the locus of Q intersects the circle with diameter OE.

$\therefore$ min. value of $QO \times QE = 168 \times 2 = 336$

Method 2

Let R_2 be the radius of the circumcircle of ΔOEQ.

area of ΔOEQ:

$$168 = \frac{OE \times QO \times QE}{4R_2}$$

$$QO \times QE = \frac{168 \times 4R_2}{30} = 22.4R_2$$

R_2 is minimum when the circumcircle of ΔOEQ is smallest, i.e. the chord OE is a diameter of the circle.

Note that the locus of Q intersects the circle with diameter $OE,$ as the distance between the locus of Q and the line segment $OE \le OF = 14 < 15 = \frac{1}{2}OE$.

$\therefore$ min. value of $R_2 = \frac{30}{2} = 15$

$\therefore$ min. value of $QO \times QE = 22.4 \times 15 = 336$