ST. PAUL'S COLLEGE

FORM 6 INTERNAL EXAMINATION 2021 - 2022

MATHEMATICS Compulsory Part

PAPER 1

Section B

Question-Answer Book

2¼ hours

This paper must be answered in English.

B

Name			
Class	( )		
Group	G1 FBL	G2 LMW	G3 WHP
	G4 TH	G5 PSK	G6 LTN
	G7 HL		

INSTRUCTIONS

1. Write your Name, Class and Class number in the spaces provided on the right. Circle your Group Number.

2. This paper consists of THREE sections, A(1), A(2) and B.

3. Attempt **ALL** questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

4. Graph paper and supplementary answer sheets will be supplied on request. Write your Name, Class and Class number in the spaces provided, mark the question number box, and fasten them with string INSIDE this book.

5. Unless otherwise specified, all working must be clearly shown.

6. Unless otherwise specified, numerical answers should be either exact or correct to 3 significant figures.

7. The diagrams in this paper are not necessarily drawn to scale.

Question No.	Marks
15	
16	
17	
18	
19	
Total	

SECTION B (35 marks)

15. There are 2 white balls, 3 yellow balls and 10 black balls in a bag. If 4 balls are drawn from the bag randomly at the same time, find the probability that
    (a) Four balls of the same colour are drawn. (2 marks)
    (b) The balls drawn are of two different colours and there are two balls of each colour. (3 marks)

Solution:

(a) The probability that four balls of the same colour are drawn

$$=\frac{C_4^{10}}{C_4^{15}}$$ 1M for denominator

$$=\frac{2}{13} \quad (\approx 0.154)$$ 1A

(b) The required probability

$$=\frac{C_2^2C_2^3+C_2^2C_2^{10}+C_2^3C_2^{10}}{C_4^{15}}$$ For the numerator, 1M for 3 cases + 1M for correct cases

$$=\frac{61}{455} \quad (\approx 0.134)$$ 1A

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16. Let $G(n)$ be the *n*th term of a geometric sequence. It is given that $G(5)=192$ and $G(4)=48$.

(a) Find $G(1)$. (2 marks)

(b) If the difference between the sum of the first *n* terms and the sum of the first 2*n* terms of the sequence $G(n)$ is less than 1.5×10^{22}, find the greatest possible value of *n*. (4 marks)

Solution:

(a) Let *a* be the first term and *r* be the common ratio.

$$\begin{cases} ar^4 = 192 \\ ar^3 = 48 \end{cases}$$ 1M for either

$a=\frac{3}{4}$ and $r=2$

$G(1)=\frac{3}{4}$ 1A

(b) $$\frac{3}{4}\left(\frac{4^{2n}-1}{4-1}\right)-\frac{3}{4}\left(\frac{4^n-1}{4-1}\right)<1.5\times10^{22}$$ 1M for $S(\infty)$

$$4^{2n}-4^n-6\times10^{22}<0$$

$$\frac{1-\sqrt{1^2+4(1)(6\times10^{22})}}{2}<4^n<\frac{1+\sqrt{1^2+4(1)(6\times10^{22})}}{2}$$ 1M

$$\log 4^n<\log\frac{1+\sqrt{1^2+4(1)(6\times10^{22})}}{2}$$ 1M for taking log (cannot be absorbed)

$$n\log 4<\log\frac{1+\sqrt{1^2+4(1)(6\times10^{22})}}{2}$$

$$n<\frac{\log\frac{1+\sqrt{1^2+4(1)(6\times10^{22})}}{2}}{\log 4}$$

$$\frac{\log\frac{1+\sqrt{1^2+4(1)(6\times10^{22})}}{2}}{\log 4}\approx18.4$$

Greatest $n=18$ 1A f.t.

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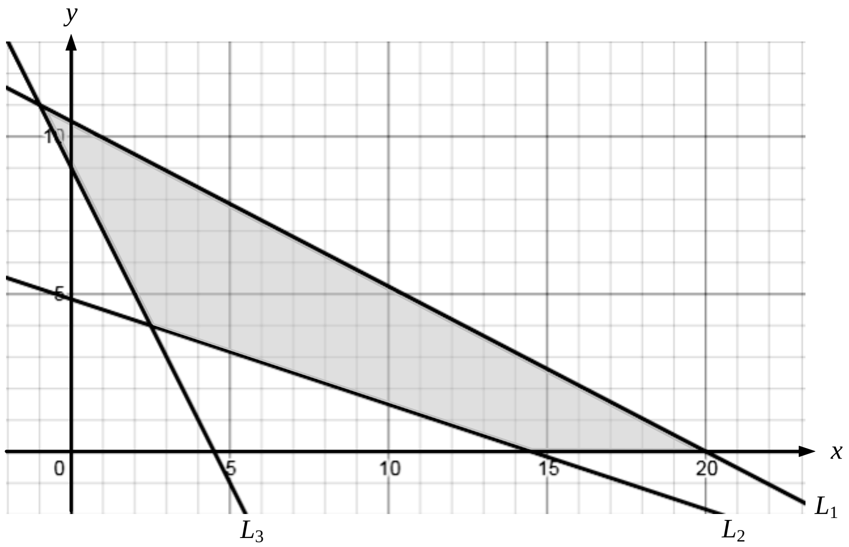
17. The shaded region R is bounded by the x-axis and the straight lines L_1, L_2 and L_3.

The equations of L_1, L_2 and L_3 are $11x+21y-220=0$, $2x+6y-29=0$ and $2x+y-9=0$ respectively.

(a) It is given that R represents the solution of a system of inequalities. Write down the system of inequalities. (2 marks)

(b) Find the least value of $7x+5y$, where (x, y) is an integral point (with x and y being integers) lying in R. (3 marks)



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Solution:

(a) The inequalities are $\begin{cases} 11x+21y-220\le 0 \\ 2x+6y-29\ge 0 \\ 2x+y-9\ge 0 \\ y\ge 0 \end{cases}$ 1A for either + 1A for all

(b) Let $P=7x+5y$

At $A(-1, 11)$, $P = 48$

At $B(20, 0)$, $P = 140$

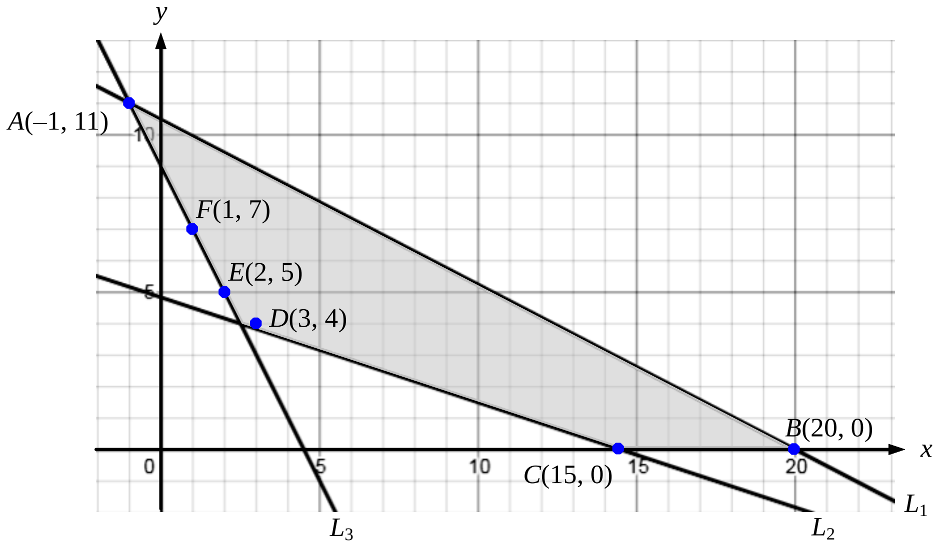
At $C(15, 0)$, $P = 105$ 1M for testing A, B and C

At $D(3, 4)$, $P = 41$

At $E(2, 5)$, $P = 39$ 1M for $D(3, 4)$ or $E(2, 5)$

At $F(1, 7)$, $P = 42$

Least value = $\underline{\underline{39}}$ 1A f.t. (must test A to E)



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18. (a) Figure 18(a) shows a piece of paper card $ADBC$ in the form of a quadrilateral with $AB = 40$ cm and $AC = 21$ cm. It is also given that $\angle ADB = 86°$, $\angle DBC = 89°$ $\angle BCA = 110°$ and $\angle BAD = 35°$. Find the length of AD. (2 marks)

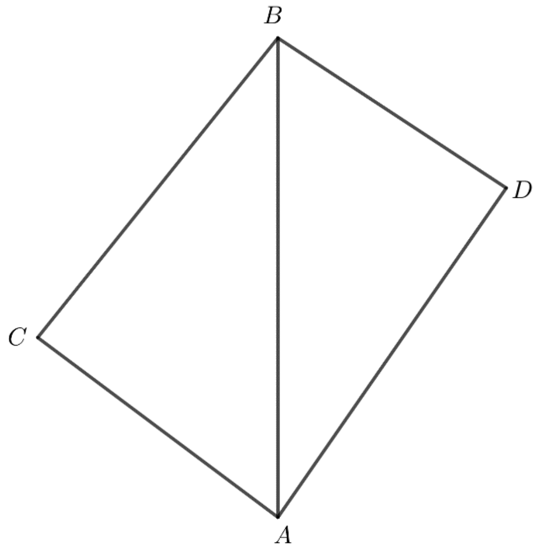


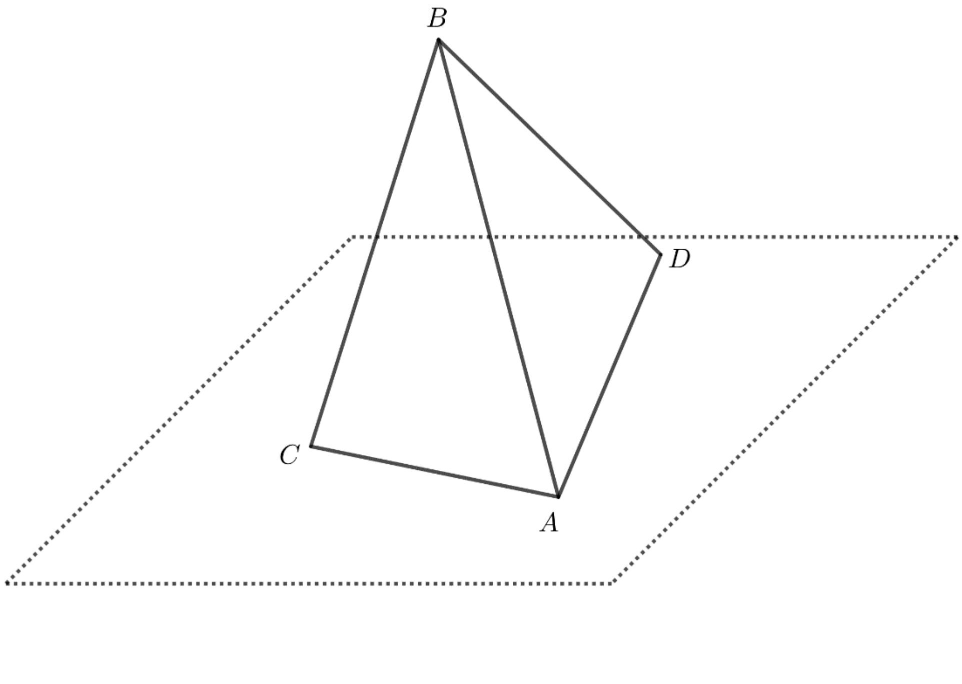
Figure 18(a)

(b) The paper card described in (a) is folded along AB such that AC and AD lie on the horizonal ground as shown in figure 18(b). The distance between C and D on the horizontal ground is 30 cm.

(i) Find $\angle CAD$.

(ii) Someone claims that the angle between the plane ABC and the plane ABD is greater than $105°$. Do you agree? Explain your answer.

(5 marks)



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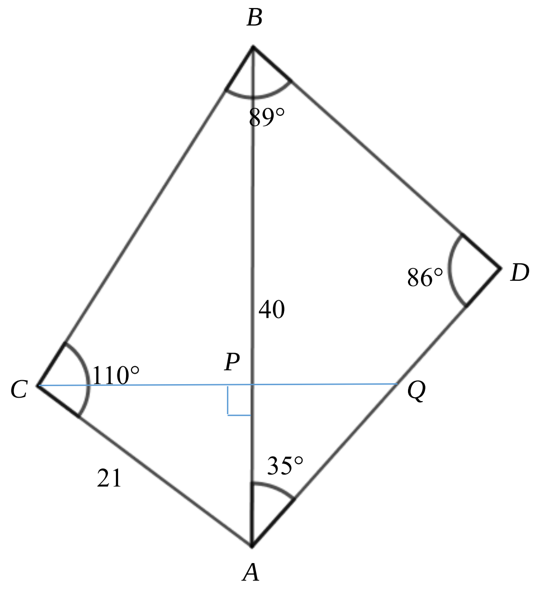
Solution

(a) $\dfrac{AD}{\sin(180°-35°-86°)}=\dfrac{40}{\sin 86°}$ 1M

$AD \approx 34.37041664$

$AD \approx 34.4$ cm 1A



(b) (i) $CD^2 = AC^2 + AD^2 - 2(AC)(AD)\cos\angle CAD$ 1M

$30^2 = 21^2 + 34.37041664^2 - 2(21)(34.37041664)\cos\angle CAD$

$\angle CAD \approx 59.97493697°$ 1A

(ii) Let P be the foot of perpendicular from C to BA. Produce CP to meet AD at Q.
The required angle is $\angle CPQ$.

$\angle CAB = 360° - 110° - 89° - 86° - 35° = 40°$

$CP = 21\sin 40°$ 1M (for either 2*)

$AP = 21\cos 40°$

$PQ = AP\tan 35° = 21\cos 40°\tan 35°$

$AQ = \dfrac{AP}{\cos 35°} = \dfrac{21\cos 40°}{\cos 35°}$

$CQ^2 = AC^2 + AQ^2 - 2(AC)(AQ)\cos\angle CAD$ 1M (for either 2*)

$CQ^2 = 21^2 + \left(\dfrac{21\cos 40°}{\cos 35°}\right)^2 - 2(21)\left(\dfrac{21\cos 40°}{\cos 35°}\right)\cos 59.97493697°$

$CQ \approx 20.34576399$

$CQ^2 = PC^2 + PQ^2 - 2(PC)(PQ)\cos\angle CPQ$

$20.34576399^2 = (21\sin 40°)^2 + (21\cos 40°\tan 35°)^2$
$\quad - 2(21\sin 40°)(21\cos 40°\tan 35°)\cos\angle CPQ$

$\angle CPQ \approx 110.1703374° > 105°$

I agree. 1A f.t.

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Alternatively,

Let θ be the required angle.

By Dihedral Angle Formula,

$$\cos\theta = \frac{\cos 59.97493697° - \cos 35° \cos 40°}{\sin 35° \sin 40°}$$

$\theta \approx 110.1703374° > 105°$

I agree. 3A f.t. (no M marks)

Page total

19. Let $\mathrm{f}(x)=3x^2-6kx+12x+3k^2-6k+12$ and $\mathrm{g}(x)=-\mathrm{f}(-x)+c$ where k and c are constants. On the same rectangular coordinate system, denote the vertex of the graph of $y=\mathrm{f}(x)$ and the vertex of the graph of $y=\mathrm{g}(x)$ by P and Q respectively. It is known that the mid-point of P and Q is (0, 8).

(a) Using the method of completing the square, express the coordinates of P in terms of k. (2 marks)

(b) Write down the value of c. (1 mark)

(c) Given a circle C: $5x^2+5y^2-66x+32y-576=0$ and a straight line L: $y=mx+8$ where m is a constant with $m>1$. L cuts C at two points $A(x_A,\ y_A)$ and $B(x_B,\ y_B)$, where $x_B>x_A$.

(i) Express the x-coordinates of A and B in terms of m.

(ii) Show that the distance between A and B is $\sqrt{m^2+1}(x_B-x_A)$.

(iii) Let G be the centre of C. If the distance between A and B is 10, find the radius of the inscribed circle of ΔABG.

(iv) When the distance between A and B is 10, is it possible for Q to be the centre of the inscribed circle of ΔABG. Explain your answer.

(9 marks)

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Solution:

(a)
$$\begin{aligned}\mathrm{f}(x)&=3x^2-6kx+12x+3k^2-6k+12\\&=3(x^2-2kx+4x)+3k^2-6k+12\\&=3\left[x^2-2(k-2)x+(k-2)^2-(k-2)^2\right]+3k^2-6k+12 &\text{1M}\\&=3[x-(k-2)]^2-3(k-2)^2+3k^2-6k+12\\&=3[x-(k-2)]^2+6k\end{aligned}$$

$P=(k-2,\ 6k)$ 1A

(b) $\mathrm{g}(x)=-\mathrm{f}(-x)+c$

$Q=(2-k,\ -6k+c)$

$c=16$ 1A

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Page total

(c) $C: \ 5x^2+5y^2-66x+32y-576=0 \quad \ldots (1)$

(i) $L: \ y=mx+8 \quad \ldots (2)$

Sub (2) into (1),

$5x^2+5(mx+8)^2-66x+32(mx+8)-576=0$ 1M

$5(m^2+1)x^2+(112m-66)x=0$

$x = 0 \quad \text{or} \quad \dfrac{66-112m}{5(m^2+1)}$ 1M

$x_A=\dfrac{66-112m}{5(m^2+1)} \quad \text{and} \quad x_B=0 \qquad \left(\text{As } \dfrac{66-112m}{5(m^2+1)}<0 \quad \text{for} \quad m>1\right)$ 1A for both

(ii) $y_B=mx_B+8 \quad \text{and} \quad y_A=mx_A+8$

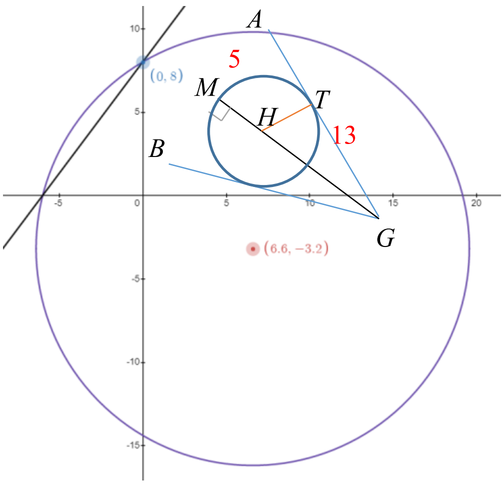
$y_B-y_A=m(x_B-x_A)$

$AB = \sqrt{(x_B-x_A)^2+(y_B-y_A)^2}$

$= \sqrt{[m(x_B-x_A)]^2+(x_B-x_A)^2}$

$= \sqrt{m^2+1}(x_B-x_A)$ 1

(iii) Let r be the radius of the inscribed circle.



$G = (6.6, -3.2)$

radius of $C=\sqrt{6.6^2+(-3.2)^2-\left(\dfrac{-576}{5}\right)}=13$

Let M be the mid-point of AB, H be the in-centre, T be the point of contact of AG and the inscribed circle and r be the radius.

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$AM = 5$

$AT = AM = 5$

$TG = 8$

$MG = \sqrt{13^2 - 5^2} = 12$

$HG = 12 - r$

$HG^2 = HT^2 + TG^2$

$(12-r)^2 = r^2 + 8^2$ 1M

$r = \dfrac{10}{3}$ 1A

(d) $\sqrt{m^2+1}\left(x_B - x_A\right) = 10$

$$\left(m^2+1\right)\left[\frac{66-112m}{5\left(m^2+1\right)}\right]^2 = 100$$

$10044m^2 - 14784m + 1856 = 0$

$2511m^2 - 3696m + 464 = 0$

$m = \dfrac{4}{3}$ or $m = \dfrac{116}{837}$ (rejected as $m > 1$) 1A

$A = (-6, 0)$

$Q = \left(2-k,\ -6k+16\right)$

$M = (-3, 4)$

If M, H and G are collinear and $MH : HG = \dfrac{10}{3} : \left(12 - \dfrac{10}{3}\right) = 5 : 13$ (from (b)(iii))

$$H = \left(\frac{13(-3)+5(6.6)}{5+13},\ \frac{13(4)+5(-3.2)}{5+13}\right) = \left(-\frac{1}{3},\ 2\right)$$ 1M for finding H

If Q is the in-centre,

$$\begin{cases} 2-k = -\dfrac{1}{3} \\ -6k+16 = 2 \end{cases}$$

$k = \dfrac{7}{3}$ satisfies both equations.

Yes, possible. 1A f.t.

End of Paper

Page total