ST. PAUL'S COLLEGE

FORM 6 INTERNAL EXAMINATION 2022 - 2023

MATHEMATICS Compulsory Part

PAPER 1

Section B

Question-Answer Book

2¼ hours

This paper must be answered in English.

B

Name	Marking Scheme
Class	( )
Group	G1 LTN G2 PSK G3 LMW G4 HL G5 YKC G6 LTN G7 HL

INSTRUCTIONS

1. Write your Name, Class and Class number in the spaces provided on the right. Circle your Group Number.
2. This paper consists of THREE sections, A(1), A(2) and B.
3. Attempt **ALL** questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.
4. Graph paper and supplementary answer sheets will be supplied on request. Write your Name, Class and Class number in the spaces provided, mark the question number box, and fasten them with string INSIDE this book.
5. Unless otherwise specified, all working must be clearly shown.
6. Unless otherwise specified, numerical answers should be either exact or correct to 3 significant figures.
7. The diagrams in this paper are not necessarily drawn to scale.

Question No.	Marks
15	
16	
17	
18	
19	
Total	

SECTION B (35 marks)

15. If 7 adults and 4 children randomly form a queue, find the probability that no children are next to each other. (3 marks)

The required probability

$= \dfrac{7! \times P_4^8}{11!}$ 1M+1M

$= \dfrac{7}{33}$ 1A

$(\approx 0.2121212121....)$

Permutation for 7 adults (7!)

A1 A2 A3 A4 A5 A6 A7

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

Out of 8 spaces, each of the 4 spaces are occupied by each children $(P_4^8 = 8 \times 7 \times 6 \times 5)$

Q15 Mean = 2.297 / 3 (76.6%)

Good. Some students confused "the probability" with "the number of permutations". Some students had difficulties in handling the number of the required permutations.

16. Let $f(x)=x^2-8kx+6x+16k^2-19k+11$, where k is a constant.

(a) Using the method of completing the square, express, in terms of k, the coordinates of the vertex of the graph of $y=f(x)$. (2 marks)

(b) On the same rectangle coordinate system, O is the origin. Let P and Q be the vertex of the graph of $y=f(x)$ and the vertex of the graph of $y=f(x-9)+6$ respectively. Is it possible that $(-9,12)$ is the orthocentre of ΔOPQ? Explain your answer. (3 marks)

(a) $f(x)$

$=x^2-8kx+6x+16k^2-19k+11$

$=x^2-2(4k-3)x+16k^2-19k+11$

$=x^2-2(4k-3)x+(4k-3)^2-(4k-3)^2+16k^2-19k+11$ 1M

$=[x-(4k-3)]^2-16k^2+24k-9+16k^2-19k+11$

$=[x-(4k-3)]^2+5k+2$

$\therefore$ The coordinates of the vertex are $(4k-3,5k+2)$ 1A

(b) Note that the coordinates of Q are $(4k+6,5k+8)$ 1M

Slope of $PQ=\dfrac{5k+8-(5k+2)}{4k+6-(4k-3)}=\dfrac{6}{9}=\dfrac{2}{3}$

Denote $(-9,12)$ by A

$(\text{Slope of } PQ)\times(\text{Slope of } OA)=\left(\dfrac{2}{3}\right)\times\left(\dfrac{-4}{3}\right)=\dfrac{-8}{9}\neq-1$ 1M

$\therefore$ OA is not perpendicular to PQ.

$\therefore$ No, it is not possible that $(-9,12)$ is the orthocentre of ΔOPQ. 1A f.t.

Alternatively,

the slope of the altitude through O with respect to the base PQ is $\dfrac{-3}{2}$.

Hence, the equation of the corresponding altitude is $y=\dfrac{-3}{2}x$.

Putting $(x,y)=(-9,12)$, L.H.S. $=12$ but R.H.S. $=\dfrac{-3}{2}(-9)=13.5\neq12$

Therefore, $(-9,12)$ does not lie on one of the altitudes of ΔOPQ.

Hence it is not possible that $(-9,12)$ is the orthocentre of ΔOPQ.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Q16 (a) Mean = 1.297 / 2 (64.9%)

(b) Mean = 1.020 / 3 (34.0%)

(a) Good. Careless mistakes were often found. Some students even did not express the coefficient of x in terms of k, and thus completed the square wrongly.

(b) Fair. Most students could give the coordinates of Q but they did not consider the slope of PQ and the corresponding altitude, which is the best way to solve the problem.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Page total

Answers written in the margins will not be marked.

17. Let $A(n)$ be the nth term of a geometric sequence, where n is a positive integer.
It is given that $A(3)=48$ and $A(6)=3072$.

(a) Find $A(n)$. (2 marks)

(b) Let $B(n)=\log A(n)$. Find the least value of k such that the sum of the first k terms of the sequence $B(n)$ is greater than 2023. (4 marks)

(a) Let a and r be the first term and the common ratio of the geometric sequence.

$$\begin{cases} ar^2=48 \\ ar^5=3072 \end{cases}$$ 1M

$$r^3=64$$
$$r=4$$
$$a(4)^2=48$$
$$a=3$$
$\therefore$ $A(n)=3(4^{n-1})$ 1A

(b) $B(n)=\log A(n)=\log\left[3(4^{n-1})\right]=\log 3+2(n-1)\log 2$ 1M log and G.S. to A.S.

$$B(1)+B(2)+...+B(k)>2023$$
$$k\log 3+2\left[1+2+...+(k-1)\right]\log 2>2023$$
$$k\log 3+2\cdot\frac{k(k-1)}{2}\log 2>2023$$ 1M sum of A.S.
$$k\log 3+k^2\log 2-k\log 2>2023$$
$$(\log 2)k^2+(\log 3-\log 2)k-2023>0$$
$$k>81.68523573...\text{ or }k<-82.27019823...$$ 1M quadratic inequality

$\therefore$ The least value of k is 82. 1A

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Page total

Alternatively,

$$B(1)+B(2)+\ldots+B(k)>2023$$
$$\log A(1)+\log A(2)+\ldots+\log A(k)>2023$$
$$\log\left[A(1)A(2)\ldots A(k)\right]>2023$$
$$3\cdot 3(4)\cdot 3(4)^2\cdot\ldots\cdot 3(4)^{k-1}>10^{2023}$$
$$3^k 4^{1+2+\ldots+k-1}>10^{2023}$$
$$3^k\cdot 4^{\frac{k(k-1)}{2}}>10^{2023}$$
$$k\log 3+\frac{k(k-1)}{2}\log 4>2023$$
$$k\log 3+k(k-1)\log 2>2023$$
$$(\log 2)k^2+(\log 3-\log 2)k-2023>0$$
$$k>81.68523573\ldots \text{ or } k<-82.27019823\ldots$$

$\therefore$ The least value of k is 82 .

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Q17 (a) Mean: 1.723 / 2 (86.1%)

(b) Mean: 1.475 / 4 (36.9%)

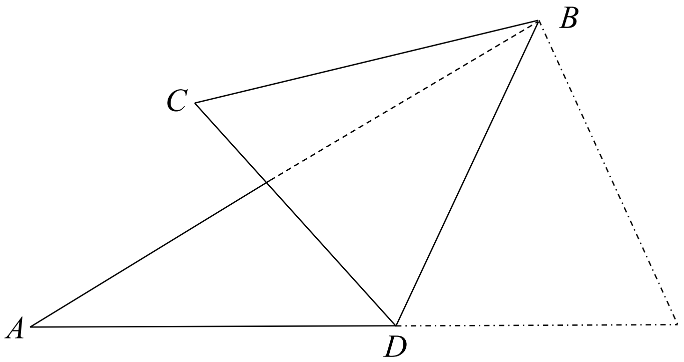
(a) Very good. Only a few students thought that $A(n)$ is an arithmetic sequence. A few students were not familiar with the general terms of geometric sequence and gave $A(n)=3(4)^n$.

(b) Fair. Not many students realized that $B(n)$ is an arithmetic sequence with first term $\log 3$ and common difference $\log 4$. Some students had difficulties in applying the formula of sum of arithmetic sequence and solving quadratic inequality.

Answers written in the margins will not be marked.

18. (a) A triangular paper card ABC with $AB = 20$ cm, $\angle ACB = 56°$ and $\angle ABC = 82°$. Find BC. (2 marks)

(b) The paper card ABC described in (a) is given. Let D be a point on AC such that BD is the angle bisector of $\angle ABC$. As shown, the paper card is folded along BD such that $AC = 10$ cm.



(i) Find $\angle ABC$.

(ii) Someone claims that the angle between ΔABD and ΔCBD is greater than $45°$. Do you agree? Explain your answer.

(6 marks)

(a) $\angle BAC + \angle ACB + \angle ABC = 180°$ ($\angle$ sum of Δ)

$\angle BAC + 56° + 82° = 180°$

$\angle BAC = 42°$

$$\frac{BC}{\sin \angle BAC} = \frac{AB}{\sin \angle ACB}$$

$$\frac{BC}{\sin 42°} = \frac{20}{\sin 56°}$$ 1M

$$BC = \frac{20 \sin 42°}{\sin 56°} \text{ cm}$$

$\approx 16.14234695...$cm

$= 16.1$ cm (cor. to 3 sig. fig.) 1A

(b) (i) Consider ΔABC in space.

$$\cos \angle ABC = \frac{AB^2 + BC^2 - AC^2}{2(AB)(BC)} \approx \frac{20^2 + (16.1423..)^2 - 10^2}{2(20)(16.1423...)}$$ 1M

$\cos \angle ABC \approx 0.868175127...$

$\angle ABC \approx 29.75273487...°$

$= 29.8°$ (cor. to 3 sig. fig.) 1A

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Page total

(b) (ii) Let E be the point lying on BD such that $CE \perp BD$.

Let F be the point lying on AB such that $EF \perp BD$.

The angle between ΔABD and ΔCBD is $\angle CEF$. 1M

We have $BF = BC \approx 16.14234695...\text{cm}$ and

$CE = FE = BC \sin 41° \approx 10.59033246...\text{cm}$.

In ΔBCF,

$CF^2 = BC^2 + BF^2 - 2(BC)(BF)\cos \angle ABC$

$CF^2 \approx (16.142..)^2 + (16.142...)^2 - 2(16.142)^2 \cos(29.75..°)$ 1M

$CF^2 \approx 68.70062835...$

$CF \approx 8.288584218...\text{cm}$

In ΔCEF,

$\cos \angle CEF = \dfrac{CE^2 + FE^2 - CF^2}{2(CE)(FE)} \approx \dfrac{(10.59...)^2 + (10.59...)^2 - (8.288...)^2}{2(10.59...)(10.59...)}$ 1M

$\cos \angle CEF \approx 0.693725016...$

$\angle CEF \approx 46.07429563...°$

$> 45°$

$\therefore$ Yes, the claim is agreed. 1A

Q18 (a) Mean: 1.762 / 2 (88.1%)

(b) (i) Mean: 1.248 / 2 (62.4%)

(ii) Mean: 0.891 / 4 (22.3%)

(a) Very good. A few students found AC instead.

(b) (i) Good. Students were recommended to make use of the memory space in calculators for carrying more significant figures throughout the working.

(ii) Poor. Not many students could identify the angle between two planes correctly. Some students used dihedral angle formula to solve the problem directly.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

19. The equation of the circle C is $x^2+y^2-400x-300y+40000=0$ and the equation of the straight line L is $y=mx$, where m is a constant. It is given that C and L intersect at two distinct points P and Q. Denote the mid-point of P and Q by M.

(a) (i) Find the range of values of m.

(ii) Show that the y-coordinate of M is $\dfrac{50m(3m+4)}{1+m^2}$.

(4 marks)

(b) Two tangents are drawn from the origin to the circle C. The points of contact are A and B respectively, where the y-coordinate of A is positive.

(i) Find the equation of the perpendicular bisector of AB.

(ii) Find the equation of the perpendicular bisector of BM in terms of m. Hence, or otherwise, find the equation of the circle passing through A, B and M.

(iii) If m varies in the range in (a)(i), denote the locus of the moving point M by Γ. Someone claims that the length of Γ is greater than 320. Do you agree? Explain your answer.

(9 marks)

(a) (i) $x^2+y^2-400x-300y+40000=0$

$x^2+(mx)^2-400x-300mx+40000=0$

$(1+m^2)x^2-(300m+400)x+40000=0$

$\Delta=\left[-(300m+400)\right]^2-4(1+m^2)(40000)$ 1M

$=10000\left[(3m+4)^2-16-16m^2\right]$

$=10000(9m^2+24m+16-16-16m^2)$

$=10000(-7m^2+24m)$

$=10000m(-7m+24)$

$\Delta>0$ 1M

$10000m(-7m+24)>0$

$0<m<\dfrac{24}{7}$ 1A

(a) (ii) Denote (x_1,y_1) and (x_2,y_2) by P and Q respectively.

x-coordinate of $M=\dfrac{x_1+x_2}{2}=\dfrac{1}{2}\cdot\dfrac{-\left[-(300m+400)\right]}{1+m^2}=\dfrac{50(3m+4)}{1+m^2}$

Since M lies on $y=mx$,

y-coordinate of $M=m\cdot\dfrac{50(3m+4)}{1+m^2}=\dfrac{50m(3m+4)}{1+m^2}$ 1

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Page total

(b) (i) Perpendicular bisector of A and B is the straight line passing through O and the centre of C. 1M

Centre of $C = \left(\frac{-(-400)}{2}, \frac{-(-300)}{2}\right) = (200, 150)$

The required equation:

$$y = \frac{150-0}{200-0}x$$

$$y = \frac{3}{4}x$$ 1A

(b) (ii) Take $m = 0$, coordinates of $B = (200, 0)$.

Coordinates of $M = \left(\frac{50(3m+4)}{1+m^2}, \frac{50m(3m+4)}{1+m^2}\right)$

Mid-point of $BM = \left(\frac{25(3m+4)}{1+m^2} + 100, \frac{25m(3m+4)}{1+m^2}\right)$

Slope of $BM = \dfrac{\frac{50m(3m+4)}{1+m^2}}{\frac{50(3m+4)}{1+m^2} - 200} = \frac{m(3m+4)}{3m+4-4-4m^2} = \frac{m(3m+4)}{m(3-4m)} = \frac{3m+4}{3-4m}$

Required equation of perpendicular bisector:

$$y - \frac{25m(3m+4)}{1+m^2} = \frac{4m-3}{3m+4}\left(x - \frac{25(3m+4)}{1+m^2} - 100\right)$$ 1M

$$y = \frac{4m-3}{3m+4}x - \frac{25(4m-3)}{1+m^2} - \frac{100(4m-3)}{3m+4} + \frac{25m(3m+4)}{1+m^2}$$

$$y = \frac{4m-3}{3m+4}x + \frac{75m^2+75}{1+m^2} - \frac{100(4m-3)}{3m+4}$$

$$y = \frac{4m-3}{3m+4}x + 75 - \frac{100(4m-3)}{3m+4}$$

$$y = \frac{4m-3}{3m+4}x + \frac{75(3m+4) - 100(4m-3)}{3m+4}$$

$$y = \frac{4m-3}{3m+4}x + \frac{-175m+600}{3m+4}$$

$$y = \frac{4m-3}{3m+4}x + \frac{25(-7m+24)}{3m+4}$$ 1M

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Page total

Putting $y=\frac{3}{4}x$

$$\frac{3}{4}x=\frac{4m-3}{3m+4}x+\frac{25(-7m+24)}{3m+4}$$
$$3(3m+4)x=4(4m-3)x+100(-7m+24)$$
$$(9m+12)x=(16m-12)x+100(-7m+24)$$
$$(9m+12-16m+24)x=100(-7m+24)$$
$$(-7m+24)x=100(-7m+24)$$
$$x=100$$

$y=\frac{3}{4}(100)=75$

Centre of the required circle $=(100,75)$

Radius of the required circle $=\sqrt{(100-200)^2+(75-0)^2}=125$ 1M

Equation of the required circle:

$(x-100)^2+(y-75)^2=125^2$ 1A

$x^2+y^2-200x-150y=0$

Alternatively,

we have $O=(0,0)$, $A=(56,192)$, $B=(200,0)$.

Denote the centre of C by G , we have $G=(200,150)$.

One could prove that O,A,M and G are concyclic. ($\angle OAG=\angle OAG=90°$)

Similarly, O,B,M and G are concyclic. (converse of angle in the same segment)

Hence, O,A,B,M and G are concyclic.

The circle passing through A,B and M is the circle passing through A,B and G.

The circle passing through A,B and G could be found by some other means.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Page total

(b) (iii) Denote the centre of C by G.

Denote the centre of circumcircle of A, B and M by D.

$\angle ADB$

$= 2\angle BDG$

$= 2\left(2\cos^{-1}\frac{100}{125}\right)$ 1M

$= 4\cos^{-1}\frac{4}{5}$

$\approx 147.4795906...^\circ$

The length of Γ

$= \frac{\angle ADB}{360^\circ} \times 2\pi(125)$ 1M

$\approx 321.7505544...$

> 320

$\therefore$ Yes, the claim is agreed. 1A f.t.

Answers written in the margins will not be marked.

Q19 (a) (i) Mean: 1.713 / 3 (57.1%)

(ii) Mean: 0.406 / 1 (40.6%)

(b) (i) Mean: 0.614 / 2 (30.7%)

(ii) Mean: 0.218 / 4 (5.4%)

(iii) Mean: 0.050 / 3 (1.7%)

(a) (i) Good. Careless mistakes like $(mx)^2 = mx^2$ and $(-300m-400)^2 = 90000m^2 + 160000$ were found. Some students could not solve the quadratic inequality correctly and gave $m > 0$ or $m < \frac{24}{7}$.

(ii) Fair. Some students could not complete the proof because of the careless mistakes in (i).

(b) (i) Fair. Most students did not realize that the perpendicular bisector of AB is the straight line through O and the centre of C.

(ii) Very poor.

(iii) Very poor.

Section B Mean: 14.71 / 35 (42.0%)

End of Paper

Answers written in the margins will not be marked.

Page total