SECTION B (35 marks)

15. A group of 5 people are randomly selected from 5 boys and 4 girls.

(a) Find the probability that a group of exactly 2 boys and 3 girls are selected. (2 marks)

(b) Find the probability that a group consisting of at most 2 boys are selected. (2 marks)

Solution:

(a) The required probability $= \dfrac{C_2^5 C_3^4}{C_5^9}$

$= \dfrac{20}{63}$ 1M +1A

(b) The required probability $= \dfrac{(C_1^5 C_4^4 + C_2^5 C_3^4)}{C_5^9}$

$= \dfrac{5}{14}$ 1M+1A

16. The first term of a geometric sequence is 16 and the sum of the second term and the third term of the sequence is 60. It is given that all terms of the sequence are positive.

(a) Find the common ratio of the sequence. (2 marks)

(b) Find the least value of n such that the sum of the $(n+1)^{\text{th}}$ term to $(2n+1)^{\text{th}}$ term of the sequence is greater than 4.8×10^{12}. (3 marks)

Solution :

(a) Let r be the common ratio of the sequence.

$16r + 16r^2 = 60$ 1M

$4r^2 + 4r - 15 = 0$

$(2r - 3)(2r + 5) = 0$

$r = 1.5$ or $r = -2.5$ (*rejected*) 1A

$\therefore$ The common ratio of the sequence is 1.5.

(b) $\dfrac{16(1.5^{2n+1} - 1)}{1.5 - 1} - \dfrac{16(1.5^n - 1)}{1.5 - 1} > 4.8 \times 10^{12}$ 1M

$(1.5)(1.5^n)^2 - (1.5^n) - 1.5 \times 10^{11} > 0$

$1.5^n > \dfrac{1 - \sqrt{(-1)^2 - 4(1.5)(-1.5 \times 10^{11})}}{2(1.5)}$ or $1.5^n > \dfrac{1 + \sqrt{(-1)^2 - 4(1.5)(-1.5 \times 10^{11})}}{2(1.5)}$ 1M

$1.5^n < -316227.4327$ (*rej.*) or $1.5^n > 316228.0994$

$n \log 1.5 > \log (316228.0994)$

$n > 31.23380733$

$\therefore$ The least value of n is 32. 1A f.t.

17. (a) Let a and b be real constants. If the roots of the equation $x^2 + ax + b = 0$ are $2p$ and $5p$, prove that $10a^2 = 49b$. (2 marks)

(b) It is given that the x-axis is a tangent to the circle C: $x^2 + y^2 - 8x + 12y + r = 0$, where r is a real constant. Find the real constant m such that the straight line $y = mx$ cuts C at the points Q and R with $OQ : OR = 2 : 5$, where O is the origin. (4 marks)

Solution:

(a) $2p + 5p = -a$

$p = -\frac{a}{7}$ ....(1)

$(2p)(5p) = b$ 　1M

$10p^2 = b$ .... (2)

Substitute (1) into (2).

$10\left(-\frac{a}{7}\right)^2 = b$

$10a^2 = 49b$ 　1A

(b) Coordinates of the centre of C

$= \left(-\frac{-8}{2}, -\frac{12}{2}\right)$

$= (4, -6)$

Note that the radius of C is 6. 　1M

$\sqrt{4^2 + (-6)^2 - r} = 6$

$52 - r = 36$

$r = 16$

Let $2t$ be the x-coordinate of Q.

Then the x-coordinate of R is $5t$.

Substitute $y = mx$ into $x^2 + y^2 - 8x + 12y + 16 = 0$.

$x^2 + (mx)^2 - 8x + 12(mx) + 16 = 0$

$(1 + m^2)x^2 + (12m - 8)x + 16 = 0$ 　1M

$x^2 + \frac{12m-8}{1+m^2}x + \frac{16}{1+m^2} = 0$

$\therefore$ $2t$ and $5t$ are the roots of the equation $x^2 + \frac{12m-8}{1+m^2}x + \frac{16}{1+m^2} = 0$.

By (a), we have $10\left(\frac{12m-8}{1+m^2}\right)^2 = 49\left(\frac{16}{1+m^2}\right)$. 　1M

$10(3m - 2)^2 = 49(1 + m^2)$

$41m^2 - 120m - 9 = 0$

$m = 3$ *or* $-\frac{3}{41}$ 　1A

18. In Figure 3(a), the diagonals of cardboard *ABCD* intersect at *E*. It is given that $AB = AD = 50$ cm, $BC = DC = 62$ cm and $\angle ABD = 53°$.

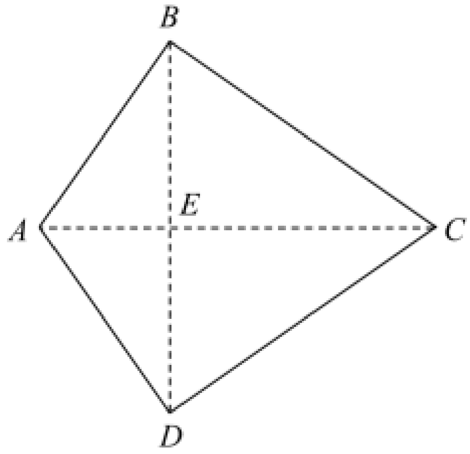

Figure 3(a)

(a) Find $\angle BCD$.

(2 marks)

(b) The cardboard *ABCD* in Figure 3(a) is folded along *BD* such that *A* is vertically above the line *EC*. Two extra triangular cardboards are placed to form the tetrahedron *ABCD* as shown in Figure 3(b). It is given that the total surface area of the tetrahedron *ABCD* is 4335 cm^3.

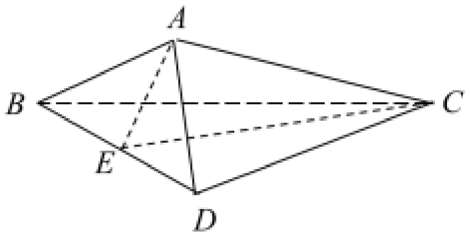

Figure 3(b)

(i) Does the angle between $\triangle ABD$ and $\triangle BCD$ exceed 20°? Explain your answer.

(ii) Find the volume of the tetrahedron *ABCD*.

(7 marks)

Solution:

(a) Consider $\triangle ABD$.

$\angle ADB = \angle ABD = 53°$ (base ∠s, isos. △)

$\angle BAD = 180° - 53° - 53°$ (∠ sum of △)

$= 74°$

By the sine formula, we have

$$\frac{BD}{\sin\angle BAD} = \frac{AD}{\sin\angle ABD}$$

$$\frac{BD}{\sin 74°} = \frac{50\text{ cm}}{\sin 53°}$$

$$BD \approx 60.18150232\text{ cm}$$

Consider $\triangle BCD$.

By the cosine formula, we have

$$\cos\angle BCD = \frac{BC^2 + CD^2 - BD^2}{2(BC)(CD)}$$

$$\approx \frac{62^2 + 62^2 - 60.18150232^2}{2(62)(62)}$$

$$\approx 0.528900465$$

for either one 1M

$$\angle BCD \approx 58.06880622°$$

$= \underline{\underline{58.1°}}$ (cor. to 3 sig. fig.) 1A

(b) (i) The area of $\triangle BCD$

$$\approx \frac{1}{2}(62)(62)\sin 58.06880622°\text{ cm}^2$$

$$\approx 1631.170383\text{ cm}^2$$ 1A

Note that $\triangle ADC \cong \triangle ABC$ (SSS).

Total surface area of the tetrahedron

4335 = area of $\triangle ABD$ + area of $\triangle BCD$ + area of $\triangle ADC$ + area of $\triangle ABC$

$$4335 = \left[\frac{1}{2}(50)(50)\sin 74° + 1631.170383 + 2(A_{\Delta ADC})\right]\text{ cm}^2$$

$$A_{\Delta ADC} = 751.1262487\text{ cm}^2$$

Let H be a point lying on DC such that AH is perpendicular to DC.

$$\frac{62 \cdot AH}{2} = 751.1262487$$

$$AH = 24.22987899$$ 1M

$$\sin\angle ADC = \frac{AH}{50}$$

$$\angle ADC = 28.9861103°$$

Consider $\triangle ADC$.

By the cosine formula, we have

$$AC^2 = (50)^2 + (62)^2 - 2(50)(62)\cos 28.9861103°$$ 1A

$$AC \approx 30.34187373\text{ cm}$$

Consider $\triangle ABE$.

$$AE = 50\sin 53° \text{ cm}$$
$$\approx 39.9317755 \text{ cm}$$

Consider $\triangle BEC$.

$$EC = \sqrt{BC^2 - BE^2} \quad \text{(Pyth. theorem)}$$
$$\approx \sqrt{62^2 - \left(\frac{60.18150232}{2}\right)^2} \text{ cm}$$
$$\approx 54.2083637 \text{ cm}$$

Consider $\triangle AEC$.

By Cosine Formula,

$$AC^2 = (AE)^2 + (CE)^2 - 2(AE)(CE)\cos$$ 1M

$$\angle AEC \approx 33.44371916° > 20°$$

$\therefore$ The angle between $\triangle ABD$ and $\triangle BCD$ exceeds 20° 1A f.t.

(Alternative Method)

Consider $\triangle AEC$.

Let $s = \dfrac{AE + AC + EC}{2} \approx 62.24100646$ cm

By the Heron's formula, we have

Area of $\triangle AEC$

$$= \sqrt{s(s-AE)(s-AC)(s-EC)} \text{ cm}^2$$
$$\approx 596.484644 \text{ cm}^2$$

Let G be a point lying on EC such that AG is perpendicular to EC. 1M

The angle between $\triangle ABD$ and $\triangle BCD$ is $\angle AEG$.

$$\frac{EC \bullet AG}{2} \approx 596.484644 \text{ cm}^2$$
$$AG = 22.00710751 \text{ cm}$$

Consider $\triangle AEG$.

$$\sin \angle AEG = \frac{AG}{AE}$$
$$\approx \frac{22.00710751}{50\sin 53°}$$
$$\angle AEG \approx 33.44372439° > 20°$$

$\therefore$ The angle between $\triangle ABD$ and $\triangle BCD$ exceeds 20° 1A f.t.

(ii) Volume of the tetrahedron

$\approx \frac{1}{3}(1631.170383)(50\sin 53^\circ)(\sin 33.4437^\circ)\text{cm}^3$ 1M

$= \underline{\underline{11965.78066 \text{ cm}^3}}$ 1A

19. The coordinates of the points A and B are (0 , 9) and (32 , −15) respectively. C is a point in the rectangular coordinate plane such that BC is a vertical line. H is a point in the same rectangular coordinate plane such that $\angle AHB = \angle AHC$ and $BH = CH$.

(a) Prove that $\triangle ABH \cong \triangle ACH$. (2 marks)

(b) Find the coordinates of C. (2 marks)

(c) Find the coordinates of the circumcentre of $\triangle ABC$. (2 marks)

(d) Suppose that $AH = BH = CH$. Denote the in-centre of $\triangle ABC$ by I.

(i) Are A, H and I collinear? Explain your answer.

(ii) Someone claims that the area of the circumcircle of $\triangle ABC$ is greater than 4 times the area of the inscribed circle of $\triangle ABC$. Do you agree? Explain your answer.

(5 marks)

Solution:

(a) In $\triangle ABH$ and $\triangle ACH$,

$\angle AHB = \angle AHC$ (*given*)

$BH = CH$ (*given*)

$AH = AH$ (*common side*)

$\therefore$ $\triangle ABH \cong \triangle ACH$ (*SAS*)

Marking Scheme:	
Case 1 Any correct proof with correct reasons.	2
Case 2 Any correct proof without reasons.	1
	---------- (2)

(b) $\because$ BC is a vertical line.

$\therefore$ x-coordinate of $C = 32$

Let (32 , k) be the coordinates of C. Note that $k > 0$.

$\because$ $\triangle ABH \cong \triangle ACH$

$\therefore$ $AB = AC$

$$\sqrt{(32-0)^2+(-15-9)^2} = \sqrt{(32-0)^2+(k-9)^2}$$ 1M

$$(k-9)^2 = 576$$

$$k-9 = 24 \text{ or } -24$$

$$k = 33 \text{ or } -15 \text{ (rejected)}$$

$\therefore$ The coordinates of C are (32 , 33). 1A

(c) Let (p , q) be the coordinates of H.

$\because$ BC is a vertical line and $BH = CH$.

$\therefore$ By symmetry, $q = 9$

Coordinates of the mid-point M of AB

$= \left(\frac{0+32}{2}, \frac{9+(-15)}{2}\right)$

$= (16 , -3)$

$\because$ $MH \perp AB$

$\therefore$ Slope of $MH \times$ slope of $AB = -1$

$$\frac{9-(-3)}{p-16} \times \frac{9-(-15)}{0-32} = -1$$

$$p = 25$$ 1M

$\therefore$ The coordinates of H are $(25 , 9)$. 1A

(d)(i) Note that H is the circumcentre of $\triangle ABC$.

$\because$ $\triangle ABH \cong \triangle ACH$

$\therefore$ $\angle BAH = \angle CAH$

$\therefore$ AH is the angle bisector of $\angle BAC$. 1M

$\therefore$ I lies on AH.

$\therefore$ A, H and I are collinear. 1A f.t.

(ii) Let R and r be the radii of the circumcircle and the inscribed circle of $\triangle ABC$ respectively.

$R = AH$
$= 25 - 0$
$= 25$

Suppose that the inscribed circle of $\triangle ABC$ touches AC and BC at D and E respectively.

$\because$ A, H and I are collinear and $AH \perp BC$.

$\therefore$ AI is a horizontal line and $AI \perp BC$.

i.e. AE is a horizontal line.

$\therefore$ The coordinates of E are (32 , 9).

$AE = 32 - 0 = 32$

$BC = 33 - (-15) = 48$ 1A

$CE = \frac{1}{2}BC = 24$

$AC = \sqrt{(32-0)^2 + (33-9)^2} = 40$

$CD = CE = 24$

$AD = AC - CD = 40 - 24 = 16$

Note that $\triangle ACE \sim \triangle AID$.

$$\therefore \quad \frac{CE}{ID} = \frac{AE}{AD}$$

$$\frac{24}{r} = \frac{32}{16}$$

$$r = 12$$

$$\frac{\text{Area of the circumcircle}}{\text{Area of the inscribed circle}} = \frac{\pi R^2}{\pi r^2} = \frac{\pi(25)^2}{\pi(12)^2} = \frac{625}{144} > 4$$ 1M

$\therefore$ The area of the circumcircle is greater than 4 times the area of the inscribed circle.

$\therefore$ The claim is agreed. 1A f.t.

(Alternative Method 1) Let R and r be the radii of the circumcircle and the inscribed circle of $\triangle ABC$ respectively. $R = AH = 25$ $AI = AE - EI = 32 - r$ In $\triangle ADI$, $DI^2 + AD^2 = AI^2$ $r^2 + 16^2 = (32 - r)^2$ $r^2 + 256 = 1024 - 64r + r^2$ $64r = 768$ $r = 12$	1A
$\dfrac{\text{Area of the circumcircle}}{\text{Area of the inscribed circle}} = \dfrac{\pi R^2}{\pi r^2} = \dfrac{\pi(25)^2}{\pi(12)^2} = \dfrac{625}{144} > 4$ $\therefore$ The area of the circumcircle is greater than 4 times the area of the inscribed circle.	1M
$\therefore$ The claim is agreed.	1A f.t.

(Alternative Method 2) Let R and r be the radii of the circumcircle and the inscribed circle of $\triangle ABC$ respectively. $R = AH = 25$ Area of $\triangle ABC$ = area of $\triangle ABI$ + area of $\triangle ACI$ + area of $\triangle BCI$ $\frac{1}{2} \times BC \times AE = \frac{1}{2} \times AB \times r + \frac{1}{2} \times AC \times r + \frac{1}{2} \times BC \times r$ $48 \times 32 = 40r + 40r + 48r$ $128r = 1\ 536$ $r = 12$	1A
$\frac{\text{Area of the circumcircle}}{\text{Area of the inscribed circle}} = \frac{\pi R^2}{\pi r^2}$ $= \frac{\pi(25)^2}{\pi(12)^2}$ $= \frac{625}{144} > 4$	1M
$\therefore$ The area of the circumcircle is greater than 4 times the area of the inscribed circle. $\therefore$ The claim is agreed.	1A f.t
(Alternative Method 3) $\because$ the circumradius is at least twice the inradius	1A
$\therefore \frac{\text{Area of the circumcircle}}{\text{Area of the inscribed circle}} = \frac{\pi R^2}{\pi r^2}$ $> \frac{\pi(2r)^2}{\pi r^2}$ $= 4$	1M
$\therefore$ The area of the circumcircle is greater than 4 times the area of the inscribed circle. $\therefore$ The claim is agreed.	1A f.t.