ST. STEPHEN'S GIRLS' COLLEGE
Final Examination 2017 - 18

Form 6 **LC, WMC, KAL, SCHL, CYN**

126 students

Mathematics
Paper I
Time allowed : 2¼ hours
Question/Answer Paper

SOLUTION

Please read the following instructions very carefully.

1. Write your class, class number, name and division (if applicable) in the spaces provided on this cover.

2. This paper consists of THREE sections, A(1), A(2) and B. Each section carries 35 marks.

3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question/Answer Paper.

4. Graph paper and supplementary answer sheets will be supplied on request. Write your class, class number and name on each sheet, and fasten them with string INSIDE this paper.

5. Unless otherwise specified, all working must be clearly shown.

6. Unless otherwise specified, numerical answers should either be exact or correct to 3 significant figures.

7. The diagrams in this paper are not necessarily drawn to scale.

Class	
Class No.	
Name	
Division	

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Total	

SECTION A(1) (35 marks)

1. Make q the subject of the formula $p = \dfrac{r-2q}{q+r}$. (3 marks)

$$p = \frac{r-2q}{q+r}$$
$$p(q+r) = r-2q$$
$$pq + pr = r-2q$$
$$pq + 2q = r - pr$$
$$q(p+2) = r - pr$$
$$q = \frac{r-pr}{p+2}$$

2. Simplify $\dfrac{x^{-8}}{(x^2y^4)^{-3}}$ and express your answer with positive indices. (3 marks)

$$\frac{x^{-8}}{(x^2y^4)^{-3}}$$
$$= \frac{x^{-8}}{x^{-6}y^{-12}}$$
$$= x^{-8-(-6)}y^{0-(-12)}$$
$$= \frac{y^{12}}{x^2}$$

3. Factorize

(a) $9a^2 - 6ab + b^2$,

(b) $9a^2 - 6ab + b^2 - 66a + 22b$.

(3 marks)

(a) $9a^2 - 6ab + b^2 = (3a-b)^2$

(b) $9a^2 - 6ab + b^2 - 66a + 22b = (3a-b)^2 - 22(3a-b)$

$= (3a-b)(3a-b-22)$

4. The prices of an apple and an orange are \$3 and \$4.5 respectively. If Susan buys 25 of these fruits for \$90, find the number of oranges bought. (4 marks)

Let x be the number of oranges bought.
Then the number of apples bought is $25 - x$.

$$\begin{aligned} 3(25 - x) + 4.5x &= 90 \\ 1.5x &= 15 \\ x &= 10 \end{aligned}$$

$\therefore$ The number of oranges bought is 10.

5. (a) Solve the inequality $2(x - 5) < 3x + 8$.
(b) Write down the least integer satisfying the inequality $2(x - 5) < 3x + 8$.
(3 marks)

(a) $2x - 10 < 3x + 8$
$-10 - 8 < 3x - 2x$
$x > -18$
(b) The required least integer is -17.

6. In Figure 1, the coordinates of point A are $(-2, 6)$. B is the image of A after it has been reflected with respect to the y-axis and then translated 4 units downwards.

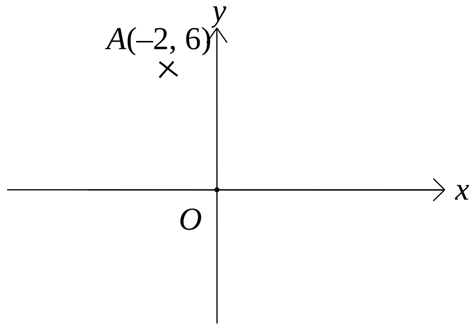


Figure 1

(a) Write down the coordinates of B.

(b) Is AB perpendicular to OB? Explain your answer.

(c) Find the area of ΔOAB.

(5 marks)

(a) The coordinates of B are $(2, 2)$.

(b) Slope of $AB = \frac{6-2}{-2-2}$

$= -1$

Slope of $OB = \frac{2}{2}$

$= 1$

$\because$ Slope of $AB \times$ Slope of $OB = -1$

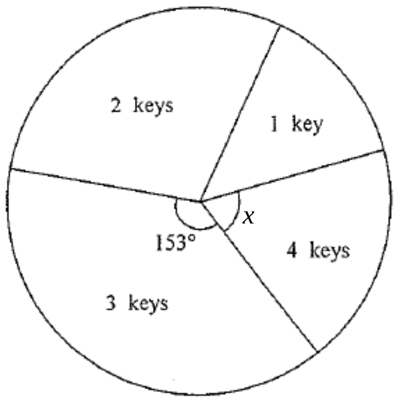
$\therefore AB$ is perpendicular to OB.

(c) Area of $\Delta OAB = AB \times OB \div 2$

$= 4\sqrt{2} \times 2\sqrt{2} \div 2$

$= 8$

7. The pie chart below shows the distribution of the numbers of keys owned by the students in a school. The numbers of students having 2 keys, 3 keys and 4 keys are 60, 85 and k respectively. If a student is randomly selected from the school, then the probability that the selected student has 4 keys is $\dfrac{7}{40}$.



(a) Find x and k.
(b) Write down the number of students in the school.

(4 marks)

(a) $x = \dfrac{7}{40} \times 360° = 63°$

$\dfrac{k}{85} = \dfrac{63}{153}$

$k = 35$

(b) The number of students in the school = 200

$(= 85 \times \dfrac{360}{153})$

8. It is given that y varies inversely as the cube root of x. When $x = 125$, $y = 35$.
   (a) Express y in terms of x.
   (b) If x is decreased by 27.1%, write down the percentage change of y.

(5 marks)

(a) Let $y = \frac{k}{\sqrt[3]{x}}$, where k is a non-zero constant.

$$35 = \frac{k}{\sqrt[3]{125}}$$

$$k = 175$$

$$\therefore y = \frac{175}{\sqrt[3]{x}}$$

(b) The percentage change

$$= \frac{\frac{175}{\sqrt[3]{0.729x}} - \frac{175}{\sqrt[3]{x}}}{\frac{175}{\sqrt[3]{x}}} \times 100\%$$

$$= 11\frac{1}{9}\%$$

9. Using an electronic balance, Peter finds that the weight of 50 identical paper clips is 79.0 g, correct to the nearest 0.1 g.
   (a) Estimate the weight of each paper clip.
   (b) Find the percentage error of the estimation in (a).
   (c) Write down the least possible weight of each paper clip.

(5 marks)

(a) Weight of each paper clip = 79.0 ÷ 50 = 1.58 g

(b) Maximum absolute error = 0.05 ÷ 50 = 0.001 g

Percentage error = $\frac{0.001}{1.58} \times 100\%$ or $\frac{0.05}{79.0} \times 100\%$

= 0.0633%

(c) 1.579 g

SECTION A(2) (35 marks)

10. In Figure 2, B is a point on AC such that $OA = OB$ and $\angle AOB = \angle OCA$.

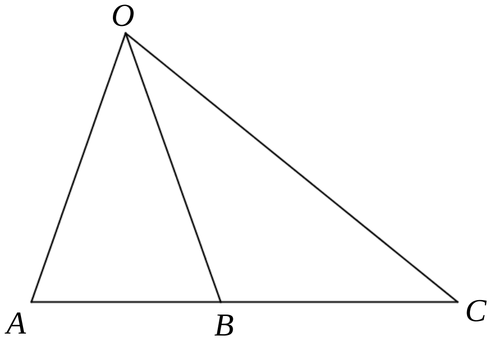


Figure 2

(a) Prove that $\Delta AOC \sim \Delta BAO$. (2 marks)

(b) It is given that $OC = 9$ cm and $AB = 4$ cm.

(i) Find OA.

(ii) If O is the centre of a circle which passes through A and B, is it possible that the area of the sector OAB is less than 11 cm^2? Explain your answer. (4 marks)

(a)

$OA = OB$

$\therefore \angle OAB = \angle OBA$ (base $\angle$s, isos. Δs)

$\therefore \angle OAC = \angle ABO$

$\angle OCA = \angle AOB$ (given)

$\boxed{\therefore \angle AOC = \angle BAO \text{ (}\angle\text{ sum of }\Delta\text{)}}$ (can be skipped)

$\therefore \Delta AOC \sim \Delta BAO$ (AAA) (or AA)

(b)(i)

$\Delta AOC \sim \Delta BAO$

$$\therefore \frac{AO}{4\text{ cm}} = \frac{9\text{ cm}}{AO}$$

$OA = 6$ cm

(ii) Area of ΔOAB

$$= \frac{1}{2} \times 4 \times \sqrt{6^2 - \left(\frac{4}{2}\right)^2}\text{ cm}^2$$

≈ 11.3 cm^2

> 11 cm^2

The area of sector $OAB >$ the area of ΔOAB

$\therefore$ It is not possible that the area of the sector OAB is less than 11 cm^2.

<u>Alternative Method</u>

Area of sector OAB $= \pi(6)^2 \times \sin^{-1}\left(\frac{2}{6}\right) \times 2 \div 360°$

≈ 12.2 cm^2

> 11 cm^2

$\therefore$ It is not possible that the area of the sector OAB is less than 11 cm^2.

11. The stem-and-leaf diagram below shows the distribution of a data set *A*. It is known that in the distribution of the data set *A*, the mean, the median and the mode are all equal.

Distribution of data set *A*

Stem (ten units)	Leaf (1 unit)
0	1 4
1	0 1 z 6 8
2	0 0 1 x x 2 3 9
3	3 7 8
4	3

(a) Find x and z.
(b) Find the range and the inter-quartile range.
(c) It is given that the range of another data set *B* is 36 and the inter-quartile range of data set *B* is 16. Determine which data set is more dispersed.

(7 marks)

(a) Median $= 21$

$\because$ Mode $=$ median $= 21$

$\therefore$ $x = \underline{\underline{1}}$

Mean

$$= \frac{1+4+10+11+(10+z)+16+18+20\times 2 + 21\times 3+22+23+29+33+37+38+43}{19}$$

$$= \frac{398+z}{19}$$

$\because$ Mean $=$ median $= 21$

$\therefore$ $\dfrac{398+z}{19} = 21$

$z = \underline{\underline{1}}$

(b) Range $= 43 - 1 = \underline{\underline{42}}$

$Q_1 = 11$

$Q_3 = 29$

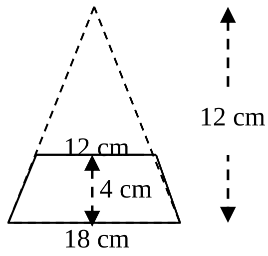
Inter-quartile range $= 29 - 11 = \underline{\underline{18}}$

(c) $\because$ Both the range and the inter-quartile range of data set *A* are greater than those of data set *B*.

$\therefore$ The data set *A* has a larger dispersion.

12. The upper base and the lower base of a right frustum X are squares with sides 12 cm and 18 cm respectively. It is also known that the height of frustum X is 4 cm.
    (a) Find the volume of frustum X. (3 marks)
    (b) There is another frustum Y which is similar to frustum X. The ratio of the volume of frustum X to the volume of frustum Y is 8 : 27 . Find the total surface area of frustum Y. (4 marks)

(a) Since 12 : 18 = 2 : 3,



The volume of frustum X

$= \frac{1}{3}\left(18^2 \times 4 \times 3 - 12^2 \times 4 \times 2\right)$ cm^3

$= 912$ cm^3

(b) The height of one lateral face of frustum $X = \frac{1}{3} \times \sqrt{12^2 + \left(\frac{18}{2}\right)^2} = 5$ cm

Total surface area of frustum X

$= 12^2 + 18^2 + 4 \cdot \frac{(12+18)(5)}{2} = 768$ cm^2

Total surface area of frustum Y

$= 768 \times \frac{9}{4} = 1728$ cm^2

13. In Figure 3, $A(0, 4)$, $B(6, 7)$ and $C(0, -2)$ are three points on the rectangular coordinate plane. L is the perpendicular bisector of AB and it intersects AB at M.

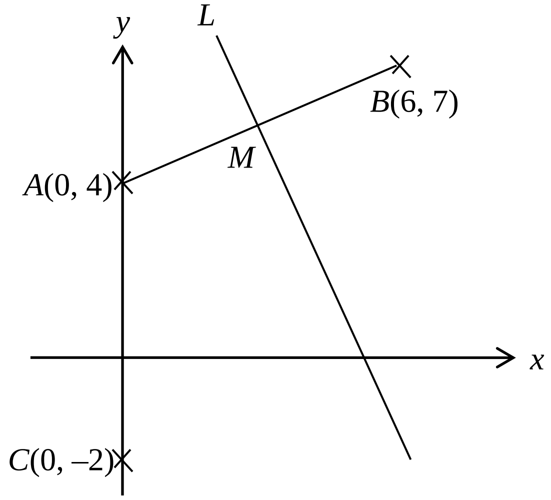


Figure 3

(a) (i) Find the slope of AB.
(ii) Hence, find the equation of L.

(3 marks)

(b) P is a point on the rectangular coordinate plane such that $PA = PB = PC$. Using (a)(ii), find the coordinates of P. (2 marks)

(c) Let H be a moving point on the circle passing through A, B and C. Using (b), find the shortest possible distance between H and M correct to 2 decimal places. (2 marks)

(a) (i) Slope of $AB = \dfrac{7-4}{6-0} = \dfrac{1}{2}$

(ii) Slope of $L = -\dfrac{1}{\left(\frac{1}{2}\right)}$

$= -2$

Coordinates of $M = \left(\dfrac{6+0}{2}, \dfrac{7+4}{2}\right)$

$= \left(3, \dfrac{11}{2}\right)$

Equation of L :

$$y - \frac{11}{2} = -2(x-3)$$
$$2y - 11 = -4x + 12$$
$$4x + 2y - 23 = 0$$

(b) P should lie on the perpendicular bisector of AC .

y–coordinate of $P = \dfrac{4+(-2)}{2} = 1$

Sub. $y = 1$ into $4x + 2y - 23 = 0$, we have

$x = \dfrac{21}{4}$

$\therefore$ Coordinates of $P = \left(\dfrac{21}{4}, 1\right)$

(c) The required distance $= PA - PM$

$= \sqrt{36.5625} - \sqrt{25.3125}$

$= 1.02$

14. Let $f(x)=1-9x-10x^2+(ax)^3$. When $f(x)$ is divided by $x+(ax)^2-2$, the quotient and the remainder are $2x-3$ and $bx+c$ respectively, where a, b and c are non-zero constants.

(a) Find a. (3 marks)

(b) Let g(x) be a quadratic polynomial such that when g(x) is divided by $x+(ax)^2-2$, the reminder is $-bx-c$.

(i) Prove that $f(x)+g(x)$ is divisible by $x+(ax)^2-2$.

(ii) Someone claims that all the roots of the equation $f(x)+g(x)=0$ are rational numbers. Do you agree? Explain your answer. (5 marks)

(a) Consider

$$a^3x^3-10x^2-9x+1=(a^2x^2+x-2)(2x-3)+bx+c$$

Compare the coefficient of x^3 term

$$a^3=2a^2$$

$\therefore$ $a=2$ $\boxed{\text{or } a=0 \text{ (rej.)}}$

(b) (i) Let g(x) = $k(a^2x^2+x-2)-bx-c$, where k is a non-zero constant.

Then,

$$f(x)+g(x)=(a^2x^2+x-2)(2x-3)+k(a^2x^2+x-2)$$
$$=(a^2x^2+x-2)(2x-3+k)$$

Thus, $f(x)+g(x)$ is divisible by $x+(ax)^2-2$.

(ii) by (a) and (b), we have $f(x)+g(x)=(4x^2+x-2)(2x-3+k)$

Consider

$$(4x^2+x-2)(2x-3+k)=0$$

Note that the roots of $4x^2+x-2=0$ are $x=\dfrac{-1\pm\sqrt{33}}{8}$ which are irrational numbers.

Thus, the claim is disagreed.

SECTION B (35 marks)

15. Let a and b be constants. Denote the graph of $y = \log_a x + b$ by G. The x-intercept of G is 16 and G passes through the point (64, 2). Express x in terms of y. (4 marks)

$$\begin{cases} 0 = \log_a 16 + b \\ 2 = \log_a 64 + b \end{cases}$$

$$\begin{cases} a^{-b} = 16 \\ a^{2-b} = 64 \end{cases}$$

$$\begin{cases} a^{-b} = 16 \\ 16(a^2) = 64 \end{cases}$$

$\therefore$ $a = 2$, $b = -4$

$\therefore$ $y = \log_2 x - 4$

$x = 2^{y+4}$

Alternate method:

$$\begin{cases} 0 = \log_a 16 + b --- (1) \\ 2 = \log_a 64 + b --- (2) \end{cases}$$

(2) – (1): $2 = \log_a \left(\frac{64}{16}\right)$

$\therefore$ $a = 2$, $b = -4$

16. A country adopts a plan to import crude oil from another country. It is given that the volume of crude oil imported in the 1st year since the start of the plan is 8×10^7 m^3 and in subsequent years, the volume of crude oil imported each year is 10% less than the volume of crude oil imported in the previous year.

(a) Find the total volume of crude oil imported in the first 10 years since the start of the plan. (2 marks)

(b) Find the least value of n such that the total volume of crude oil imported in the first n years since the start of the plan exceeds 7×10^8 m^3. (2 marks)

(a) The required volume

$= 8 \times 10^7 + 8 \times 10^7 \times 0.9 + \ldots + 8 \times 10^7 \times 0.9^9$

$= \dfrac{8 \times 10^7 (1 - 0.9^{10})}{1 - 0.9}$

≈ 521057247.9 m^3

$\approx 5.21 \times 10^8$ m^3

(b) $\dfrac{8 \times 10^7 (1 - 0.9^n)}{1 - 0.9} > 7 \times 10^8$

$8(1 - 0.9^n) > 7$

$0.9^n < 0.125$

$n \log 0.9 < \log 0.125$

$n > 19.7$

The least value of n is 20.

17. A jury of 5 is chosen at random from 6 men and 6 women. Find the probabilities that

(a) there are 2 men and 3 women in the jury, (2 marks)
(b) there are at least 4 women in the jury, (2 marks)
(c) there are more men than women in the jury. (2 marks)

(a) The required probability $= \dfrac{C_2^6 \times C_3^6}{C_5^{12}}$

$= \dfrac{25}{66}$

(b) The required probability $= \dfrac{C_1^6 \times C_4^6}{C_5^{12}} + \dfrac{C_5^6}{C_5^{12}}$

$= \dfrac{4}{33}$

(c) The required probability $= 1 - \dfrac{25}{66} - \dfrac{4}{33}$

$= \dfrac{1}{2}$

OR

The required probability $= \dfrac{C_2^6 \times C_3^6}{C_5^{12}} + \dfrac{C_1^6 \times C_4^6}{C_5^{12}} + \dfrac{C_5^6}{C_5^{12}}$

$= \dfrac{1}{2}$

OR

The required probability $= \dfrac{1}{2}$ (by symmetry)

18. The equation of the parabola Γ is $y = -x^2 + mx$, where m is a real constant. Denote the straight line $y = x + 9$ by L.
    (a) If the points of intersection of L and Γ are $A(a, p)$ and $B(b, q)$, express $(a - b)^2$ in terms of m. (3 marks)
    (b) If L touches Γ at the point Q,
        (i) find the value(s) of m by using the result in (a),
        (ii) find the coordinates of Q.

(5 marks)

(a) Putting $y = x + 9$ in $y = -x^2 + mx$, we have
$-x^2 + mx = x + 9$
$x^2 + (1 - m)x + 9 = 0$ … (*)
Note that the roots of the equation $x^2 + (1 - m)x + 9 = 0$ are a and b.
So, we have $a + b = m - 1$ and $ab = 9$.
$(a - b)^2$
$= a^2 - 2ab + b^2$
$= (a + b)^2 - 4ab$
$= (m - 1)^2 - 4(9)$
$= m^2 - 2m - 35$

(b) (i) $m^2 - 2m - 35 = 0$
$(m - 7)(m + 5) = 0$
$m - 7 = 0$ or $m + 5 = 0$
$m = 7$ or -5

(ii) When $m = 7$, (*) becomes $x^2 - 6x + 9 = 0$.
$(x - 3)^2 = 0$
$x - 3 = 0$
$x = 3$
$Q = (3, 12)$
When $m = -5$, (*) becomes $x^2 + 6x + 9 = 0$.
$(x + 3)^2 = 0$
$x + 3 = 0$
$x = -3$
$Q = (-3, 6)$

19. Figure 4(a) shows a piece of square paper *ABCD* of length 10 cm. *E* and *F* are points on *BC* and *DC* respectively such that *AF* bisects $\angle DAC$ and *AE* bisects $\angle BAC$. *B'* and *D'* are points on *BC* and *DC* respectively such that $BE = EB' = D'F = FD$. It is given that *AC* intersects *EF* and *B'D'* at *I* and *J* respectively.

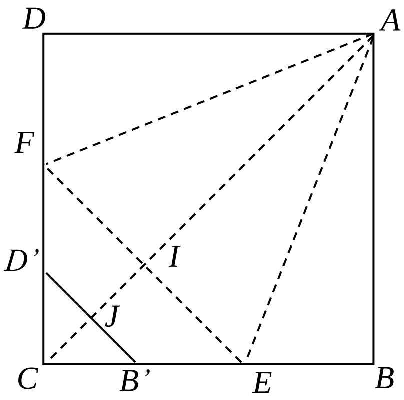


Figure 4(a)

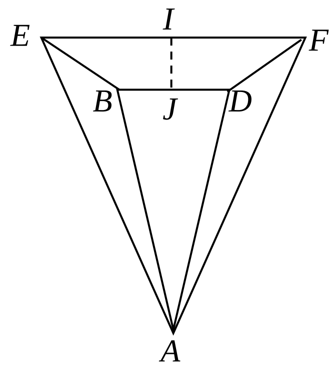


Figure 4(b)

(a) Find the length of *DF*. (2 marks)

(b) Find the area of *EB'D'F* and the length *IJ*. (5 marks)

(c) A piece of pentagonal paper *ABB'D'D* is formed by cutting off the triangular part *B'CD'* from the square paper *ABCD*. The pentagonal paper is folded along *AE*, *AF* and *EF*, such that *BE* and *DF* coincide with *B'E* and *D'F* respectively to form a container as shown in Figure 4(b).

(i) Find the angle between the planes *BDFE* and *AFE*.

(ii) Find the capacity of the container.

(6 marks)

(a) $\tan\frac{90°}{4} = \frac{DF}{10}$

$DF \approx 4.1421356\text{ cm}$

(b) $FC = CE = 10 - DF \approx 5.85786\text{ cm}$

$CD' = CB' = 10 - 2\times DF \approx 1.7157\text{ cm}$

The area of *EB'D'F*

$= \frac{1}{2}\times FC\times CE - \frac{1}{2}\times CD'\times CB'$

$\approx 15.68542495\text{ cm}^2$

$D'F \approx 4.1421356\text{ cm}$

Alternative

$IE = BE \approx 4.1421356\text{ cm}$

$JB' = IE - IJ$

$JB' \approx 1.2132034\text{ cm}$

Area of *BDFE*

$= \frac{1}{2}(JB' + IE)(IJ)(2)$

$\approx 15.68542495\text{ cm}^2$

Note that $\angle CFI = 45°$

Let *M* be a point on *FE* s.t. $D'M \perp FE$. Note that $\Delta D'MF$ is a right-angled isosceles triangle.

$FM^2 + D'M^2 = D'F^2$

Note that $IJ = FM = D'M$

$\therefore 2\,IJ^2 = D'F^2$

$IJ \approx 2.928932\text{ cm}$

Alternative

$IJ = D'F\sin 45°$

$\approx 2.928932\text{ cm}$

(c) (i) Consider figure 4(b)

The required angle is $\angle JIA$.

$IF = DF \approx 4.1421356\text{ cm}$

$DJ = IF - IJ$

$DJ \approx 1.2132034\text{ cm}$

$AJ = \sqrt{AD^2 - DJ^2}$

$AJ \approx 9.926134$

AI = 10 cm

$AJ^2 = AI^2 + IJ^2 - 2(AI)(IJ)\cos\angle JIA$

$\angle JIA \approx 80.120718°$

The required angle is 80.1°

Alternative

In Figure 4(a)

$D'J = CD'\sin 45°$

≈ 1.2132034 cm

$\therefore$ In Figure 4(b), DJ = 1.2132034 cm

(ii) Volume of the container

$= \frac{1}{3}$(Area of $BDFE$)(10)$\sin\angle JIA$

$\approx 51.50944\text{ cm}^3$

End of Paper