ST. STEPHEN'S GIRLS' COLLEGE
Final Examination 2018 - 19

Form 6 **MWC, YRK, SCHL, CYN, MLY**

122 students

Mathematics
Paper I
Time allowed : 2¼ hours
Question/Answer Paper

SOLUTION

Please read the following instructions very carefully.

1. Write your class, class number, name and division (if applicable) in the spaces provided on this cover.

2. This paper consists of THREE sections, A(1), A(2) and B. Each section carries 35 marks.

3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question/Answer Paper.

4. Graph paper and supplementary answer sheets will be supplied on request. Write your class, class number and name on each sheet, and fasten them with string INSIDE this paper.

5. Unless otherwise specified, all working must be clearly shown.

6. Unless otherwise specified, numerical answers should either be exact or correct to 3 significant figures.

7. The diagrams in this paper are not necessarily drawn to scale.

Class	
Class No.	
Name	
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SECTION A(1) (35 marks)

1. Make x the subject of the formula $\frac{x+1}{x}=\frac{3}{2y}$. (3 marks)

$$\frac{x+1}{x}=\frac{3}{2y}$$
$$2xy+2y=3x$$
$$x(2y-3)=-2y$$
$$x=\frac{-2y}{2y-3} \quad \text{or} \quad x=\frac{2y}{3-2y}$$

2. Simplify $\frac{(a^{-3}b^{2})^{-2}}{a^{-5}b^{3}}$ and express your answer with positive indices. (3 marks)

$$\frac{(a^{-3}b^{2})^{-2}}{a^{-5}b^{3}}$$
$$=\frac{a^{6}b^{-4}}{a^{-5}b^{3}}$$
$$=\frac{a^{6}a^{5}}{b^{4}b^{3}}$$
$$=\frac{a^{11}}{b^{7}}$$

3. (a) Round up 2017.195 to 2 significant figures.
   (b) Round off 2017.195 to the nearest integer.
   (c) Round down 2017.195 to 2 decimal places.

(3 marks)

(a) 2100
(b) 2017
(c) 2017.19

4. A bag contains x black balls, 6 white balls and 8 green balls. If a ball is drawn from the bag at random, then the probability of getting a black ball is $\frac{3}{4}$. Find the value of x.

(3 marks)

$$\frac{x}{x+6+8}=\frac{3}{4}$$
$$x = 42$$

5. Factorize

(a) $8p^2q-4p^3$,

(b) $8p^2q-4p^3+9p-18q$.

(4 marks)

(a) $8p^2q-4p^3=4p^2(2q-p)$

(b) $8p^2q-4p^3+9p-18q$

$=4p^2(2q-p)+9p-18q$

$=4p^2(2q-p)-9(2q-p)$

$=(2q-p)(4p^2-9)$

$=\underline{\underline{(2q-p)(2p+3)(2p-3)}}$

6. (a) Find the range of values of x which satisfy both $\dfrac{25-2x}{3}\geq 2x+1$ and $3x-11<0$.

(b) Write down the greatest integer which satisfies both inequalities in (a). (4 marks)

(a)

$25-2x\geq 6x+3$

$8x\leq 22$

$x\leq 2.75$

$\therefore \quad x\leq 2.75$

(b) 2

7. The marked price of a jacket is \$880, and it is now sold at a discount of 15% on its marked price. If the percentage profit is 25%, find the cost price of the jacket. (5 marks)

Selling price of the jacket = $880(1-15\%)$
= \$748

Cost price of the jacket = $748 \div (1+25\%)$
= \$598.4

Or

Let \$$x$ be the cost price of the jacket.

$$\frac{748-x}{x} \cdot 100\% = 25\%$$

$$x = 598.4$$

8. In Figure 1, AE is the diameter of the semi-circle and $AB = BC = CD$.

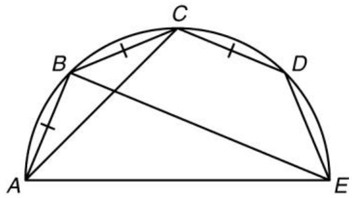


Figure 1

Let $\angle ACB = q$. Express $\angle AEB$, $\angle BAE$ and $\angle CDE$ in terms of q. (5 marks)

$\angle AEB = \angle ACB = q$ ($\angle$s in the same segment)

$\angle ABE = 90°$ ($\angle$ in semi-circle)

In $\triangle ABE$,

$\angle ABE + \angle AEB + \angle BAE = 180°$ ($\angle$ sum of Δ)

$90° + q + \angle BAE = 180°$

$\angle BAE = 90° - q$

In $\triangle ABC$,

$\because$ $AB = BC$ (given)

$\therefore$ $\angle BAC = \angle ACB = q$ (base $\angle$s, isos. $\triangle$)

$\angle CAE = \angle BAE - \angle BAC$

$\angle CAE = 90° - q - q$

$\angle CAE = 90° - 2q$

$\angle CAE + \angle CDE = 180°$ (opp. $\angle$s, cyclic quad.)

$90° - 2q + \angle CDE = 180°$

$\angle CDE = 90° + 2q$

9. Billy drives from city P to city Q at an average speed of 72 km/h and then he drives from city Q to city R at an average speed of 76 km/h. It is given that he drives 240 km in 195 minutes for the whole journey. How far does Billy drive from city P to city Q? (5 marks)

Let x km be the distance travelled from city P to city Q. Then, the distance travelled from city Q to city R is $(240 - x)$ km.

$$\frac{x}{72}+\frac{240-x}{76}=\frac{195}{60}$$

$$4x = 504$$
$$x = 126$$

Thus, Billy drives 126 km from city P to city Q.

SECTION A(2) (35 marks)

10. The stem-and-leaf diagram below shows the distribution of the heights (in cm) of a team of policemen.

Stem (tens)	Leaf (units)								
17	1	2	2	3	4	4	5	6	a
18	0	1	4	8					
19	2	b	b						

It is given that the median and the range of the above distribution are 177 cm and 24 cm respectively.

(a) (i) Find a and b.

(ii) Hence, write down the mean of the distribution. (3 marks)

(b) Four policemen join the team and two of them are 177 cm tall. It is given that the median and the mean remain unchanged. John claims that the range of the distribution remains unchanged. Do you agree with his claim? Explain your answer. (2 marks)

(a) (i) $\dfrac{176+170+a}{2}=177$

$a = 8$

$190 + b - 171 = 24$

$b = 5$

(ii) Mean = 180 cm

(b) Let the heights of the other two policemen be m cm and n cm where $m \geq n$, then $m + n = 180 \times 20 - 180 \times 16 - 177 \times 2 = 366$

Since the median remains unchanged, so $n \leq 177 \leq m$,

if $m = 196$, then $n = 170$, the new range is $(196 - 170)$ cm = 26 cm.

Thus, the claim is disagreed.

11. The table below shows the distribution of the number of water bottles owned by some students.

Number of water bottles	0	1	2	3	4
Number of students	$2n$	4	n	14	2

It is given that n is a positive even number.

(a) If the mode of the distribution is 3, write down
   (i) the least possible value of n;
   (ii) the greatest possible value of n. (2 marks)

(b) If the median of the distribution is 2, write down
   (i) the least possible value of n;
   (ii) the greatest possible value of n. (2 marks)

(c) If the mean of the distribution is 1.48, find the value of n. (2 marks)

(a) (i) 2
    (ii) 6

(b) (i) 6
    (ii) 10

(c) $$\frac{0\times 2n+1\times 4+2\times n+3\times 14+4\times 2}{2n+4+n+14+2}=1.48$$

$n = 10$

12. In Figure 2, *ABCD* is a trapezium with $\angle BCD = 90°$ and $AD // BC$. *E* is a point lying on *CD* such that $\angle AEB = 90°$.

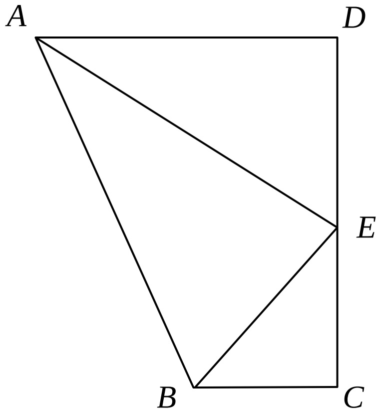


Figure 2

(a) Prove that $\Delta BCE \sim \Delta EDA$. (2 marks)

(b) It is given that BC = 9 cm, AD = 16 cm and AE = 20 cm.

(i) Find the length of *CE*.

(ii) Find the area of ΔABE.

(iii) Is there a point *F* lying on *AB* such that the distance between *E* and *F* is less than 11.5 cm? Explain your answer. (6 marks)

(a)

$\angle BCE = 90°$ (given)

$\angle EDA = 180° - \angle BCE$ (int. $\angle s$, $AD // BC$)

$\angle EDA = 90°$

$\angle EDA = \angle BCE$

$\angle BCE + \angle EBC = \angle BED$ (ext. $\angle$ of Δ)

$= \angle AEB + \angle AED$

$\because \angle AEB = 90°$ (given)

$\therefore \angle BCE = \angle AEB$

$\therefore \angle EBC = \angle AED$

$\Delta BCE \sim \Delta EDA$ (AA)

Marking Scheme:

Case 1 Any correct proof with correct reasons.

Case 2 Any correct proof without reasons.

(b) (i)

$ED = \sqrt{20^2 - 16^2}$

$= 12$ cm

$\frac{CE}{BC} = \frac{DA}{ED}$ (by (a))

$\frac{CE}{9} = \frac{16}{12}$

$CE = 12$ cm

(ii) $BE = \sqrt{12^2 + 9^2}$

$= 15 \text{ cm}$

Area of ΔABE

$= \dfrac{15 \times 20}{2}$

$= 150 \text{ cm}^2$

(iii) $AB = \sqrt{20^2 + 15^2}$

$= 25 \text{ cm}$

The shortest distance from E to AB

$= \dfrac{150 \times 2}{25}$

$= 12 \text{ cm}$

$> 11.5 \text{ cm}$

Thus there is no point F lying on AB such that the distance between E and F is less than 11.5 cm.

13. Let $f(x) = 4x^3 - 13x^2 + 40x + a$, where a is a constant. It is given that $f(x) \equiv (x-2)(bx^2 + cx + 30)$, where b and c are constants.

(a) Find a, b and c. (4 marks)

(b) How many real roots does the equation $f(x) = 0$ have? Explain your answer. (3 marks)

(a) By comparing the coefficients of x^3 and the constant terms,
we have $a = -60$ and $b = 4$.
Note that the coefficients of x^2 in the expansion of $(x-2)(bx^2 + cx + 30)$ is $c - 2b$.
By comparing the coefficients of x^2, we have $c - 2b = -13$.
(or by comparing the coefficients of x, we have $30 - 2c = 40$.)
Thus, we have $c = -5$.

<u>Alternative method</u>
Note that $x - 2$ is a factor of $f(x)$.

$$f(2) = 0$$
$$4(2)^3 - 13(2)^2 + 40(2) + a = 0$$
$$a = -60$$
$$f(x) = 4x^3 - 13x^2 + 40x - 60$$
$$= (x-2)(4x^2 - 5x + 30)$$

Thus, we have $b = 4$ and $c = -5$.

(b)
$$f(x) = 0$$
$$(x-2)(4x^2 - 5x + 30) = 0$$
$$x - 2 = 0 \text{ or } 4x^2 - 5x + 30 = 0$$

$$(-5)^2 - 4(4)(30)$$
$$= -455$$
$$< 0$$

So, the equation $4x^2 - 5x + 30 = 0$ does not have real roots.

For $x - 2 = 0$, we have $x = 2$.
Hence, the equation $(x-2)(4x^2 - 5x + 30) = 0$ has 1 real root.
Therefore, the equation $f(x) = 0$ has 1 real root.

14. A right circular cylindrical container of base radius 36 cm and height 6 cm and an inverted right circular conical vessel of base radius 20 cm and height 80 cm are held vertically. The container is fully filled with water. The water in the container is now poured into the vessel.

(a) Find the volume of water in the vessel in terms of π. (2 marks)

(b) Find the radius of water surface in the vessel. (4 marks)

(c) If a solid metal sphere of radius 13 cm is then put into the vessel and the sphere is totally immersed in the water, will the water overflow? Explain your answer. (3 marks)

(a) The volume of water in the vessel

$= \pi(36^2)(6)$

$= 7776\pi \text{ cm}^3$

(b) Let r cm be the radius of water surface in the vessel.

Then, the depth of water is $4r$ cm.

$$\frac{1}{3}\pi r^2(4r) = 7776\pi$$

$$r^3 = 5832$$

$$r = 18$$

Thus, the radius of water surface in the vessel is 18cm.

(c) The volume not occupied by water in the vessel

$=\frac{1}{3}\pi(20)^2(80) - 7776\pi$

$=\frac{8672}{3}\pi \text{ cm}^3$

The volume of the metal sphere

$=\frac{4}{3}\pi(13)^3$

$=\frac{8788}{3}\pi \text{ cm}^3$

$>\frac{8672}{3}\pi \text{ cm}^3$

Thus, the water will overflow.

SECTION B (35 marks)

15. Consider the 8 numbers 1, 2, 3, 4, 5, 6, 7, 9.
    (a) How many 4-digit numbers can be formed if each digit can be used only once? (1 mark)
    (b) How many 4-digit numbers can be formed if each digit can be used only once and the numbers formed must be odd and greater than 5000? (2 marks)

(a) The number of 4-digit numbers formed $= P_4^8$
$= 1680$

(b) The number of 4-digit number $= 3 \cdot P_2^6 \cdot 3 + 4 \cdot P_2^6 \cdot 2$
$= 510$

Or $P_2^6 \cdot 4 + P_2^6 \cdot 5 + P_2^6 \cdot 4 + P_2^6 \cdot 4$
$= 510$

16. T_1, T_2, T_3, T_4, T_5 is a geometric sequence, where all the terms are positive numbers. It is given that $\log T_2 + \log T_4 = 2$.
    (a) Find T_3. (3 marks)
    (b) Find the product of all the terms of the sequence. (2 marks)

(a) $\because$ $\log T_2 + \log T_4 = 2$
$\therefore$ $\log T_2 \times T_4 = \log 100$
$T_2 \times T_4 = 100$
$T_3^2 = 100$
$T_3 = \underline{\underline{10}}$ *or* -10 (*rejected*)

(b) $\because$ T_1, T_3, T_5 is a geometric sequence.
$\therefore$ $T_1 \times T_5 = T_3^2$
$= 100$
Product of all the terms
$= T_1 \times T_2 \times T_3 \times T_4 \times T_5$
$= (T_1 \times T_5) \times (T_2 \times T_4) \times T_3$
$= 100 \times 100 \times 10$
$= \underline{\underline{100\ 000}}$

17. Figure 3(a) shows a piece of triangular paper card ABC with AB = 36 cm, BC = 42 cm and AC = 26 cm. Let D be a point lying on AC such that $\angle ABD = 22°$.

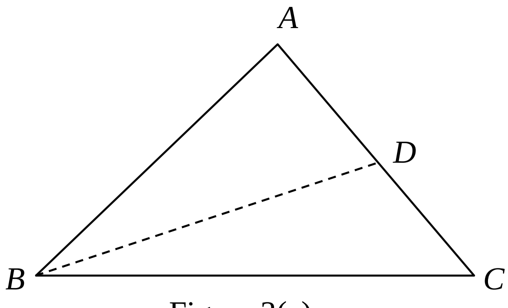


Figure 3(a)

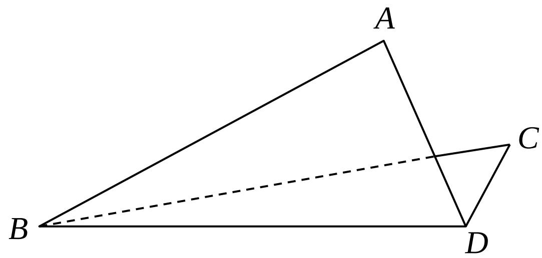


Figure 3(b)

(a) Find the length of AD. (3 marks)

(b) The paper card in Figure 3(a) is folded along BD such that the distance between A and C is 15 cm (see Figure 3(b)). Find the angle between the plane ABD and the plane BCD. (4 marks)

(a) In ΔABC,

$$\cos\angle BAC = \frac{AB^2 + AC^2 - BC^2}{2(AB)(AC)}$$

$$\cos\angle BAC = \frac{36^2 + 26^2 - 42^2}{2(36)(26)}$$

$$\angle BAC \approx 83.621° \qquad \boxed{83.62062979°}$$

In ΔABD,

$$\frac{AD}{\sin\angle ABD} = \frac{AB}{\sin\angle ADB}$$

$$\frac{AD}{\sin 22°} \approx \frac{36}{\sin(180° - 22° - 83.621°)}$$

$$AD \approx 14.0 \text{ cm} \qquad \boxed{14.00302978 \text{ cm}}$$

(b) Let P be the foot of perpendicular from A to BD in Figure 3(a). Suppose AP produced meets BC at Q.
The required angle is $\angle APQ$.

$AP = AB\sin\angle ABD$ (either one)
$= 36\sin 22°$
≈ 13.486 cm 13.48583736 cm

In Figure 3(a),

$$\cos\angle ABC = \frac{AB^2 + BC^2 - AC^2}{2(AB)(BC)}$$

$$\cos\angle ABC = \frac{36^2 + 42^2 - 26^2}{2(36)(42)}$$

$\angle ABC \approx 37.967°$ 37.96750571°

$PQ = BP\tan\angle DBC$
$= (AB\cos\angle ABD)\tan\angle DBC$
$\approx (36\cos 22°)\tan(37.968° - 22°)$
≈ 9.5507 cm 9.550681663 cm

$$BQ = \frac{BP}{\cos\angle DBC}$$

$$\approx \frac{36\cos 22°}{\cos(37.968° - 22°)}$$

≈ 34.718 cm 34.7181179 cm

In Figure 3(b),

$$\cos\angle ABC = \frac{AB^2 + BC^2 - AC^2}{2(AB)(BC)}$$

$$\cos\angle ABC = \frac{36^2 + 42^2 - 15^2}{2(36)(42)}$$

$\angle ABC \approx 20.364°$ 20.36413481°

$AQ^2 = AB^2 + BQ^2 - 2(AB)(BQ)\cos\angle ABQ$

$AQ^2 \approx 36^2 + (34.718)^2 - 2(36)(34.718)\cos 20.364°$

$AQ \approx 12.565$ cm 12.56482202 cm

$$\cos\angle APQ = \frac{AP^2 + PQ^2 - AQ^2}{2(AP)(PQ)}$$

$$\cos\angle APQ \approx \frac{(13.486)^2 + (9.5507)^2 - (12.565)^2}{2(13.486)(9.5507)}$$

$\angle APQ \approx 63.4°$ 63.43313258°

Therefore, the required angle is 63.4°

18. It is given that $f(x)$ partly varies as x^2 and partly varies as x. Suppose that $f(2) = -64$ and $f(3) = -108$.

(a) Find $f(x)$. (2 marks)

(b) Let Q be the vertex of the graph of $y = f(x)$ and R be the vertex of the graph of $y = -48 - f(x)$.

(i) Using the method of completing the square, find the coordinates of Q.

(ii) Write down the coordinates of R.

(iii) The coordinates of the point S are (–39, –72). Let P be the circumcentre of ΔQRS. Someone claims that P is the mid-point of the line segment joining Q and R. Do you agree? Explain your answer. (6 marks)

(a) Let $f(x) = ax^2 + bx$, where a and b are non-zero constants.
So, we have $4a + 2b = -64$ and $9a + 3b = -108$.
Solving, we have $a = -4$ and $b = -24$
Thus, we have $f(x) = -4x^2 - 24x$

(b) (i)

$$\begin{aligned} f(x) &= -4x^2 - 24x \\ &= -4(x^2 + 6x) \\ &= -4(x^2 + 6x + 9 - 9) \\ &= -4(x+3)^2 + 36 \end{aligned}$$

Therefore, the coordinates of Q are (–3, 36).

(ii) The coordinates of R are (–3, –84).

(iii) Note that the y-coordinate of P = y-coordinate of mid-point of RQ.
Let P be (k, –24).
Let T be the mid-point of S and R.
The coordinates of T are (–21, –78).
$\because PT \perp SR$

$$\therefore \frac{-24+78}{k+21} \times \frac{-84+72}{-3+39} = -1$$

$$\frac{54}{k+21} \times \frac{-12}{36} = -1$$

$$k = -3$$

The coordinates of P are (–3, –24).

The coordinates of the mid-point of QR

$$= \left(-3,\ \frac{-84+36}{2} \right)$$

= (–3, 24)

Thus, the claim is agreed.

Alternative 1

(iii) The slope of $QS = \dfrac{36+72}{-3+39}$

$= 3$

The slope of $RS = \dfrac{-84+72}{-3+39}$

$= -\dfrac{1}{3}$

Hence, the product of the slope of QS and the slope of RS is -1. 1M

So, $\angle QSR$ is a right angle.

Therefore, QR is a diameter of the circle which passes through Q, R and S.

Note that P is the circumcentre of ΔQRS.

Thus, the claim is agreed.

Alternative 2

(iii) $QS^2 + RS^2$

$= (-3+39)^2 + (36+72)^2 + (-3+39)^2 + (-84+72)^2$

$= 14400$

QR^2

$= (36+84)^2$

$= 14400$

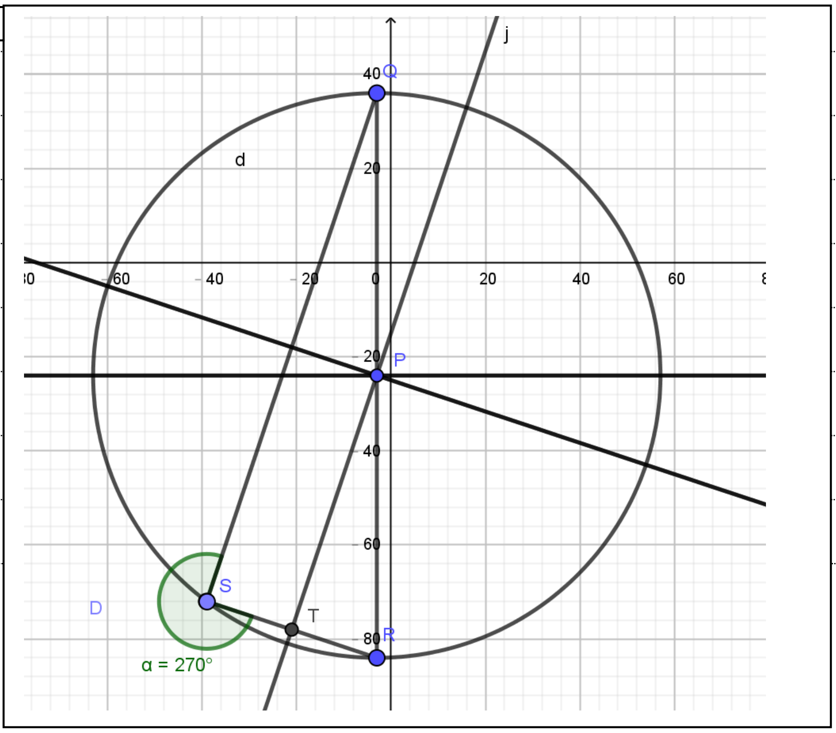
Hence, we have $QS^2 + RS^2 = QR^2$.

So, $\angle QSR$ is a right angle.

Therefore, QR is a diameter of the circle which passes through Q, R and S.

Note that P is the circumcentre of ΔQRS.

Thus, the claim is agreed.



19. The coordinates of the centre of the circle C are (3, 2). It is given that L is a tangent to C, and L cuts the x-axis and the y-axis at the points A (10, 0) and B (0, 5) respectively.

(a) Find the equation of C. (4 marks)

(b) L' is another tangent to C from A. L' cuts the y-axis at point D.

(i) Is ΔABD an obtuse-angled triangle? Explain your answer.

(ii) Find the x-coordinate of the orthocentre of ΔABD. (8 marks)

(a) Let the equation of the circle be $(x-3)^2+(y-2)^2=r^2 \ldots(1)$

Equation of L: $\dfrac{y-0}{x-10}=\dfrac{5}{-10} \Rightarrow x=10-2y \ldots(2)$

Substitute (2) into (1):

$$(10-2y-3)^2+(y-2)^2=r^2$$

$$\Rightarrow 5y^2-32y+(53-r^2)=0$$

$\Delta = 0$: $32^2-4(5)(53-r^2)=0$

$$r^2=\frac{9}{5}$$

[Alternative

$$y=5-\frac{x}{2}$$

$$(x-3)^2+(5-\frac{x}{2}-2)^2=r^2$$

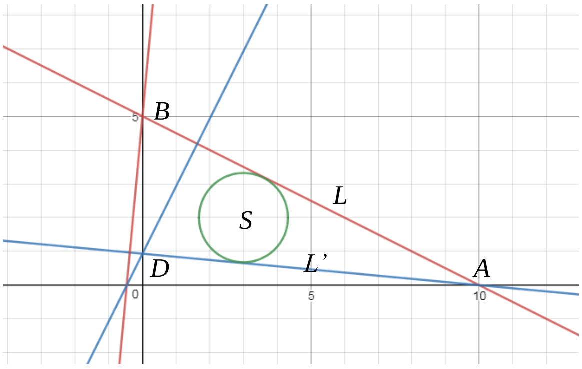
$$\Rightarrow \frac{5}{4}x^2-9x+(18-r^2)=0$$

$\Delta = 0$: $9^2-4(\frac{5}{4})(18-r^2)=0$

$$r^2=\frac{9}{5}$$

Equation of C: $\underline{(x-3)^2+(y-2)^2=\frac{9}{5}}$ or $\underline{5x^2+5y^2-30x-20y+56=0}$

F.6



(b) (i) Let O be origin. Let equation of tangents through A be $y=-\frac{c}{10}x+c$, where c is y-intercept of the tangents. Substitute in equation of C:

$$(1+\frac{c^2}{100})x^2-(\frac{c^2}{5}-\frac{2c}{5}+6)x+(c^2-4c+\frac{56}{5})=0$$

$\Delta = 0$: $$(\frac{c^2}{5}-\frac{2c}{5}+6)^2-4(1+\frac{c^2}{100})(c^2-4c+\frac{56}{5})=0$$

$$59c^2-350c+275=0$$

$$c=5\text{ (rej) or }\frac{55}{59}\Rightarrow\text{ slope of }L'=-\frac{55}{590}$$

$$\tan\angle OAD=\frac{55}{590}\Rightarrow\angle OAD=5.33^{\circ}$$

[<u>Alternative</u>:
Let S be the centre of C,

$SA=\sqrt{(10-3)^2+(0-2)^2}=\sqrt{53}$

$\sin\angle SAB=\sqrt{\frac{9}{5}}\div\sqrt{53}\Rightarrow\angle SAB=10.6^{\circ}$

$\angle BAD = 2\times\angle SAB = 21.2^{\circ}$ (tangent properties)

But $\tan\angle OAB=\frac{1}{2}\Rightarrow\angle OAB=26.6^{\circ}$

$\angle OAD = 26.6^{\circ}-21.2^{\circ}=5.33^{\circ}$

$\angle ADB = 90^{\circ}+5.33^{\circ}=95.3^{\circ}$ (ext $\angle$ of Δ)

<u>ΔABD is an obtuse-angled triangle</u>.

(ii) slope of $L'=-\frac{55}{590}$ (-0.0932)

Let the line perpendicular to L' interpolate x-axis at $x=a$

$$\frac{5-0}{0-a}\times(-\frac{55}{590})=-1\Rightarrow a=-\frac{55}{118}\ (-0.466)$$

[<u>Alternative</u>: The coordinates of D are $(0, \frac{55}{59})$,

line perpendicular to L is $y=2x+\frac{55}{59}$, $x=-\frac{55}{118}$ when $y=0$.

so the x-coordinate of the orthocentre is <u>$-\frac{55}{118}$</u>.

End of Paper